EECS-1019c: Test #2

Electrical Engineering & Computer Science

York University

Family Name: ______________________________

Given Name: ______________________________

Student#: ______________________________

- **Instructor:** Parke Godfrey
- **Exam Duration:** 75 minutes
- **Term:** Fall 2015

Answer the following questions to the best of your knowledge. Your answers may be brief, but be precise and be careful. The exam is closed-book and closed-notes. No aids such as calculators, etc., are allowed. Write any assumptions you need to make along with your answers, *if necessary*. Your answers must be legible.

There are four major questions, each with parts. Points for each question and sub-question are as indicated. In total, the test is out of 50 points.

If you need additional space for an answer, just indicate clearly where you are continuing.

Marking Box	
1.	/10
2.	/15
3.	/15
4.	/10
Total	/50

1. (10pts) **Sequences, Summations, & Infinities.**

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a. (5pts)

i. (1pt) Consider the sequence $a_i = 7 + 5i$. What is a_3?

$a_3 = 22$

ii. (1pt) Consider the sequence $b_i = 2 \cdot 3^i$. What is b_3?

$b_3 = 54$

iii. (1pt) Consider the recurrence relation

$$\begin{aligned} f_1 &= 3 \\ f_i &= 3 + 2f_{i-1} \qquad \text{for } i > 1 \end{aligned}$$

What is f_3?

$f_2 = 3 + 2 \cdot 3 = 9$
$f_3 = 3 + 2 \cdot 9 = 21$

iv. (1pt) Consider the recurrence relation

$$\begin{aligned} g_1 &= 3 \\ g_i &= \lceil g_{i-1}^2/2 \rceil \qquad \text{for } i > 1 \end{aligned}$$

What is g_3?

$g_2 = \lceil 3^2/2 \rceil = \lceil 9/2 \rceil = 5$
$g_3 = \lceil 5^2/2 \rceil = \lceil 25/2 \rceil = 13$

v. (1pt) Consider the summation

$$S(n) = \sum_{i=0}^{n} \sum_{j=0}^{i} ij$$

What is $S(3)$?

$S(3) = 25$

b. (2pts) Show that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$.

Let $S(x) = \sum_{k=0}^{\infty} x^k$.

$$\begin{aligned} S(x) &= x^0 + x^1 + x^2 + \ldots \\ &= x^0 + x(x^0 + x^1 x^2 + \ldots) \\ &= 1 + xS(x) \end{aligned}$$

$$(1-x)S(x) = 1$$

$$S(x) = \frac{1}{1-x}$$

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c. (3pts) Consider $\mathcal{A} = \{1/x \mid x \in \mathbb{Z}^+\}$. (Thus, $\mathcal{A} \subset \mathbb{Q}^+$.) Prove that $|\mathcal{A}| = |\mathbb{Z}^+|$.

Consider the function $f : \mathbb{Z}^+ \to \mathcal{A}$ *where* $f(x) = 1/x$. *For* $x, y \in \mathbb{Z}^+$ *such that* $x \neq y$, $1/x \neq 1/y$. *Thus,* f *is injective (into).* f^{-1} *over the domain* $\mathcal{A}$ *is clearly defined, (with codomain* $\mathbb{Z}^+$*):* $f^{-1}(x) = 1/x$. *For* $x, y \in \mathcal{A}$ *such that* $x \neq y$, $1/x \neq 1/y$. *Thus,* f^{-1} *is injective (into). Thus,* f *is bijective. It follows that* $|\mathcal{A}| = |\mathbb{Z}^+|$.

2. (15pts) **Algorithms & Complexity.**

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a. (5pts) For $f(n) = 7n \log_3 n$, state whether each of the following is *true* or *false*.

i. (1pt) $f(n)$ is $\mathcal{O}(n^2)$.

True.

ii. (1pt) $f(n)$ is $\Omega(n^2)$.

False.

For each of the following, state what the Θ is via a function *in the simplest form possible.*

iii. (1pt) Consider $g(n) = \dfrac{n^4}{8} + \dfrac{5n^3}{12} + \dfrac{3n^2}{8} + \dfrac{n}{12}$.
What is Θ of $g(n)$?

$g(n)$ is $\Theta(n^4)$.

iv. (1pt) Consider $S(n) = \displaystyle\sum_{i=0}^{n} i^6$.
What is Θ of $S(n)$?

$S(n)$ is $\Theta(n^7)$.

v. (1pt) Consider $S(n) = \displaystyle\sum_{i=0}^{n} (137i^5 + 67i^4 - 23i^3 - 53i^2 + 13i + 1729)$.
What is Θ of $S(n)$?

$S(n)$ is $\Theta(n^6)$.

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b. (2pts) Dr. Dogfurry tells you that he has determined that a strange function that he and you have been studying is $\mathcal{O}(\frac{1}{2}n^2 + 17)$. Briefly, what is wrong with what he has said?

$(\frac{1}{2}n^2 + 17)$ is $\mathcal{O}(n^2)$. So we would want to present the simpler form. It does not make sense to have multiplicative factors like $\frac{1}{2}$ or constants such as 17 as this is what $\mathcal{O}$ is meant to abstract away.

c. (4pts) Prove explicitly that 3^n is $\mathcal{O}(n!)$. Show the C and k that you use in your proof.

basis case*:*

$$3^7 = 2187 < 5040 = 7!.$$

inductive hypothesis*:*

Assume $3^k = k!$*, for some* $k \geq 7$*.*

inductive step*:*

$3^{k+1} = 3 \cdot 3^k$
$3^k \leq k!$ *by the inductive hypothesis.*
$3 \cdot 3^k \leq 3k!$
$3k! < (k+1)k! = (k+1)!$ *as* $k \geq 7$
$3^{k+1} < (k+1)!$

Here, $C = 1$ *and* $k = 7$*.*

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d. (4pts) Consider the following algorithm.

```
1  procedure search(x: integer, a1, a2, ..., an: distinct integers)
2  i := 1
3  while (i ≤ n and x ≠ ai)
4       i := i + 1
5  if i ≤ n then location := i
6  else location := 0
7  return location
```

Provide an argument for the big-$\mathcal{O}$ worst-case running time of the procedure *search*. (Define a function that counts the steps the algorithm takes in worst case, and then show the big-$\mathcal{O}$ of that function.)

$f(n) = 1$ *[line 2]*
$+\ n(3)$ *[lines 3 & 4]*
$+\ 1$ *[fails first condition in line 3]*
$+\ 2$ *[lines 5 & 6]*
$+\ 1$ *[line 7]*
$= 3n + 5$

We assume worst-case, so we go through the loop every time until $i = n + 1$*.* $(3n + 5)$ *is a polynomial and is* $\Theta(n)$*.*

3. (15pts) **Induction & Recursion.**

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a. (2pts) *Proof by induction.* Fill in the following blanks.

   i. (1pt) For proof by induction to be applicable, the domain over which the induction argument is to be made must have the __well-ordering__ property.

   ii. (1pt) The proof in the inductive step assumes the __inductive hypothesis__.

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b. (5pts) Prove by induction that $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$ for all integers $n \geq 1$.

Let $S(n) = \sum_{i=1}^{n} i(i!)$. *Let* $f(n) = (n+1)! - 1$.

Basis case:

$S(1) = 1(1!) = 1.$
$f(1) = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1.$

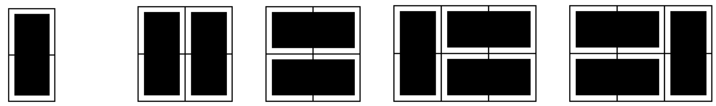
Inductive Hypothesis:

$S(k) = f(k)$ *for some* $k \geq 1$.

Inductive Step:

$$\begin{aligned}
S(k+1) &= S(k) + (k+1)(k+1)! \\
&= f(k) + (k+1)(k+1)! \qquad \textit{by inductive hypothesis} \\
&= ((k+1)! - 1) + (k+1)(k+1)! \\
&= (k+2)(k+1)! - 1 \\
&= f(k+1)
\end{aligned}$$

c. (4pts) Consider the tiling of a $(2 \times n)$ checkerboard with (2×1)-sized dominoes.



2×1 board 2×2 boards 2×3 boards ...

There is only one way to tile the (2×1) checkerboard with (2×1)-sized dominoes, two ways to tile the (2×2) checkerboard, and so forth.

Write a recurrence relation that counts correctly the number of ways to tile the $(2 \times n)$ checkerboard with (2×1)-sized dominoes.

$$d_1 = 1$$

$$d_2 = 2$$

$$d_n = d_{n-1} + d_{n-2} \qquad \textit{for } n > 2$$

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d. (4pts) Dr. Mark Dogfurry presented the following inductive proof at a conference of the conjecture, "All mongooses are the same colour."

basis case:

Consider any set of one mongoose. That mongoose has one colour. So the conjecture is trivially true.

inductive hypothesis:

Consider that, for any set of k mongooses, for some $k \geq 1$, all the mongooses in the set are the same colour.

inductive step:

Consider a set $\mathcal{M}$ of $k+1$ mongooses. Put aside one of the mongooses, call it *Fred*, from $\mathcal{M}$. The remaining set $\mathcal{M}'$ contains k mongooses. By the induction hypothesis, these are all the same colour.

Put aside a different one of the mongooses, call it *Susan*, from $\mathcal{M}$. The remaining set $\mathcal{M}''$ contains k mongooses. By the inductive hypothesis, these are all the same colour.

Fred is the same colour as the mongooses in $\mathcal{M} - \{\textit{Fred}, \textit{Susan}\}$, and *Susan* is the same colour as the mongooses in $\mathcal{M} - \{\textit{Fred}, \textit{Susan}\}$. Therefore, all the mongooses in set $\mathcal{M}$ are the same colour.

You do not believe his proof is correct. Show the flaw in his reasoning.

The proof in the inductive step is not correct for $k = 2$. Let $\mathcal{M}''' = \mathcal{M} - \{\textit{Fred}, \textit{Susan}\} = \{\}$. So there is no mongoose in $\mathcal{M}'''$ that both Fred and Susan are the same colour as.

4. (10pts) **Counting & Combinatorics.**

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a. (2pts) Automobile licence plates in Ontario usually consist of four capital letters, followed by three digits; for example, "AMJF 151". Assume that this is their only pattern.

The capital letters are the English letters A..Z (so twenty-six of them) and the digits 0..9 (so ten of them).

Assuming all possible codes of the pattern—four capital letters followed by three digits—are permitted, how many possible Ontarian licence plate codes are there? (You may write your answer in an abbreviated form such as $2^3 3^2$ rather than solving for the number.)

$$26^4 10^3 \qquad (= 456,976,000)$$

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b. (3pts) Dr. Dogfurry is designing a new protocol. To start a session, a *token* is generated that is an eight-bit string (a byte). (That is, a string of eight letters over 0's and 1's.) A token must start with "000" or with "11". The remaining five or six bits, respectively may each be either a "0" or a "1".

How many possible tokens are there in Dr. Dogfurry's protocol? Show briefly how you derive your answer *and* give the number.

Let us count the number of tokens that commence with "000": there are $2^5 = 32$ *possible ways to set the remaining five digits. Let us count the number of tokens that commence with "11": there are* $2^6 = 64$ *possible ways to set the remaining six digits. No token can start with "000"* and *with "11", thus these are disjoint. So there are* $32 + 64 = 96$ *possible tokens.*

c. (2pts) At the University Fair, three members of the York Discrete Math Club are to be chosen as speakers to give three talks at 10:00am, 10:15am, and 10:30am, respectively.

The club has eight members. How many different talk schedules—that is, choices from the members as speakers for the 10:00am, 10:15am, and 10:30am talks, respectively—are there?

Show briefly how you derive this *and* provide the number.

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$$

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d. (3pts) The UofT Discrete Math Club wants to play the York Discrete Math Club in a match of the new popular game called *Induction*. In a match of *Induction*, two teams of four people each play each other.

The UofT Club has seven members, the York Club has eight. How many different groups of eight people, four from the UofT Club and four from the York Club, for a match are possible?

Show briefly how you derive this *and* provide the number.

$$\binom{7}{4}\binom{8}{4} = \frac{7!}{4!3!} \cdot \frac{8!}{4!3!} = 35 \cdot 70 = 2,450$$

Extra Space

Relax. Turn in your exam. Return to the wild.

REFERENCE *(Detach this page for convenience, if you wish.)*

Summations

- $\displaystyle\sum_{i=0}^{n} i \quad = \quad \frac{n^2}{2} + \frac{n}{2}$
- $\displaystyle\sum_{i=0}^{n} i^2 \quad = \quad \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$
- $\displaystyle\sum_{i=0}^{n} i^3 \quad = \quad \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \quad = \quad \left(\sum_{i=0}^{n} i\right)^2$
- $\displaystyle\sum_{i=0}^{n} i^4 \quad = \quad \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$
- $\displaystyle\sum_{i=0}^{n-1} (2n+1) \quad = \quad n^2$
- $\displaystyle\sum_{k=0}^{n} ar^k \quad = \quad \frac{ar^{n+1} - a}{r - 1}$ for $r \neq 0$ and $r \neq 1$
- $\displaystyle\sum_{k=0}^{\infty} x^k \quad = \quad \frac{1}{1 - x}$ for $|x| < 1$
- $\displaystyle\sum_{k=1}^{\infty} kx^{k-1} \quad = \quad \frac{1}{(1 - x)^2}$ for $|x| < 1$

Combinatorics

- $C(n, i) = \binom{n}{i} = \dfrac{n!}{i!(n-i)!}$
- $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$

Reference

Function Growth

$\log_2 n$	n	n^2	n^3	2^n	3^n	$n!$
undef	0	0	0	1	1	1
0	1	1	1	2	3	1
1	2	4	8	4	9	2
1.58496	3	9	27	8	27	6
2	4	16	64	16	81	24
2.32193	5	25	125	32	243	120
2.58496	6	36	216	64	729	720
2.80735	7	49	343	128	2187	5040
3	8	64	512	256	6561	40320
3.16993	9	81	729	512	19683	362880
3.32193	10	100	1000	1024	59049	3628800

Asymptotic Complexity (e.g., Big-Oh)

- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\mathcal{O}(g(x))$ if there are constants C and k such that $|f(x)| \leq C|g(x)|$ whenever $x > k$. This is read as "$f(x)$ is *big-oh* of $g(x)$."
- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are constants C and k such that $|f(x)| \geq C|g(x)|$ whenever $x > k$. This is read as "$f(x)$ is *big-Omega* of $g(x)$."
- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Theta(g(x))$ if it is both $\mathcal{O}(g(x))$ and $\Omega(g(x))$. This is read as "$f(x)$ is *big-Theta* of $g(x)$."