Section 8.1 [6pt]

11. [6pt]

a. [2pt] Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

\[ a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2. \]

b. [2pt] What are the initial conditions?

\[ a_0 = 1, \ a_1 = 1. \]

c. [2pt] In how many ways can this person climb a flight of eight stairs?

\[ a_8 = f_9 = 34. \]
Section 8.2 [24pt]

4. [10pt] Solve these recurrence relations together with the initial conditions given.

a. [2pt] \(a_n = a_{n-1} + 6a_{n-2}\) for \(n \geq 2\), \(a_0 = 3\), \(a_1 = 6\).

\[
\text{Characteristic equation: } r^2 - r - 6, \text{ so roots are } -5 \text{ and } 1. \quad \alpha_1 + \alpha_2 = a_0 = 3, \\
-5\alpha_1 + \alpha_2 = a_1 = 6. \quad \text{Solving, } \alpha_1 = -\frac{1}{2}, \alpha_2 = 3\frac{1}{2}. \\
\therefore a_n = -\frac{1}{2}(-5)^n + 3\frac{1}{2}.
\]

b. [2pt] \(a_n = 7a_{n-1} - 10a_{n-2}\) for \(n \geq 2\), \(a_0 = 2\), \(a_1 = 1\).

\[
\text{Characteristic equation: } r^2 - 7r + 10, \text{ so roots are } -5 \text{ and } -2. \\
\alpha_1 + \alpha_2 = a_0 = 2, -5\alpha_1 - 2\alpha_2 = a_1 = 1. \quad \text{Solving, } \alpha_1 = -\frac{5}{3}, \alpha_2 = \frac{11}{3}. \\
\therefore a_n = -\frac{5}{3}(-5)^n + \frac{11}{3}(-2)^n.
\]

c. [2pt] \(a_n = 6a_{n-1} - 8a_{n-2}\) for \(n \geq 2\), \(a_0 = 4\), \(a_1 = 10\).

\[
\text{Characteristic equation: } r^2 - 6r + 8, \text{ so roots are } -4 \text{ and } -2. \\
\alpha_1 + \alpha_2 = a_0 = 4, -4\alpha_1 - 2\alpha_2 = a_1 = 10. \quad \text{Solving, } \alpha_1 = -9, \alpha_2 = 13. \\
\therefore a_n = -9(-4)^n + 13(-2)^n.
\]

d. [2pt] \(a_n = 2a_{n-1} - a_{n-2}\) for \(n \geq 2\), \(a_0 = 4\), \(a_1 = 1\).

\[
\text{Characteristic equation: } r^2 - 2r + 1, \text{ so just a single root of } -1 \text{ with a multiple of } 2. \\
\alpha_1 + 0\alpha_2 = a_0 = 4, -\alpha_1 - 1\alpha_2 = a_1 = 1. \quad \text{Solving, } \alpha_1 = 4, \alpha_2 = 5. \\
\therefore a_n = 4(-1)^n - 5n(-1)^n.
\]

e. [2pt] \(a_n = a_{n-2}\) for \(n \geq 2\), \(a_0 = 5\), \(a_1 = -1\).

\[
\text{Characteristic equation: } r^2 - 1, \text{ so roots are } -1 \text{ and } 1. \\
\alpha_1 + \alpha_2 = a_0 = 5, -\alpha_1 + \alpha_2 = a_1 = -1. \quad \text{Solving, } \alpha_1 = 3, \alpha_2 = 2. \\
\therefore a_n = 3(-1)^n + 2.
\]
8. [4pt] A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a. [2pt] Find a recurrence relation for \( \{L_n\} \), where \( L_n \) is the number of lobsters caught in year \( n \), under the assumption for this model.

\[
L_n = \frac{1}{2}L_{n-1} + \frac{1}{2}L_{n-2}, \text{ for } n > 2.
\]

b. [2pt] Find \( L_n \) if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

Characteristic equation is \( r^2 - \frac{1}{2}r - \frac{1}{2} = 0 \), or \( 2r^2 - r - 1 = 0 \). The roots are \( 1, -\frac{1}{2} \). Thus, we know \( \alpha_1 - \frac{1}{2}\alpha_2 = 100000 \) and \( \alpha_1 + \frac{1}{4}\alpha_2 = 300000 \). Solving, we get \( \alpha_1 = \frac{700000}{3} \) and \( \alpha_2 = \frac{800000}{3} \). \( \therefore L_n = 233333.33 + 266666.67(-\frac{1}{2})^n \).

12. [4pt] Find the solution to \( a_n = 2a_{n-1} + a_{n-2}2a_{n-3} \) for \( n = 3, 4, 5, \ldots \), with \( a_0 = 3 \), \( a_1 = 6 \), and \( a_2 = 0 \).

The characteristic equation is \( r^3 - 2r^2 - r + 2 = 0 \). The roots are \(-1, 1, \) and \( 2 \). Thus, \( \alpha_1 + \alpha_2 + \alpha_3 = a_0 = 3, -\alpha_1 + \alpha_2 + 2\alpha_3 = a_1 = 6, \) and \( \alpha_1 + \alpha_2 + 4\alpha_3 = a_2 = 0 \). Solving, \( \alpha_1 = -2, \alpha_2 = 6, \) and \( \alpha_3 = -1 \). \( \therefore a_n = -2(-1)^n + 6 - (2)^n \).

24. [6pt] Consider the nonhomogeneous linear recurrence relation \( a_n = 2a_{n-1} + 2^n \).

a. [2pt] Show that \( a_n = 2^{n+1} \) is a solution of this recurrence relation.

Solving the corresponding homogeneous, the characteristic equation is \( r^2 - 2r = 0 \), with a root of 2. Thus, \( a_n^{(h)} = \alpha 2^n \).

Because \( f(n) = 2^n \), since \( 2^n \) appears in our homogeneous solution with a multiplicity of 1, a reasonable trial solution is \( a_n^{(p)} = cn2^n \). Thus, \( cn2^n = 2c(n-1)2^{n-1} + 2^n = c(n-1)2^n + 2^n = cn2^n - c2^n + 2^n \), so \( c = 1 \). Hence, a particular equation is \( a_n^{(p)} = n2^n \).

So, the full solution is \( a_n = a_n^{(h)} + a_n^{(p)} = \alpha 2^n + n2^n = (\alpha + n)2^n \).

(Cannot show what the question wants; it is not correct.)

b. [2pt] Use Theorem 5 to find all solutions of this recurrence relation.

All solutions by Theorem 5 are of the form \( a_n = (\alpha + n)2^n \), as established above.

c. [2pt] Find the solution with \( a_0 = 2 \).

\( (\alpha + 0)2^0 = \alpha = 2 \). Thus, \( \alpha = 2 \) and \( a_n = (2 + n)2^n \).