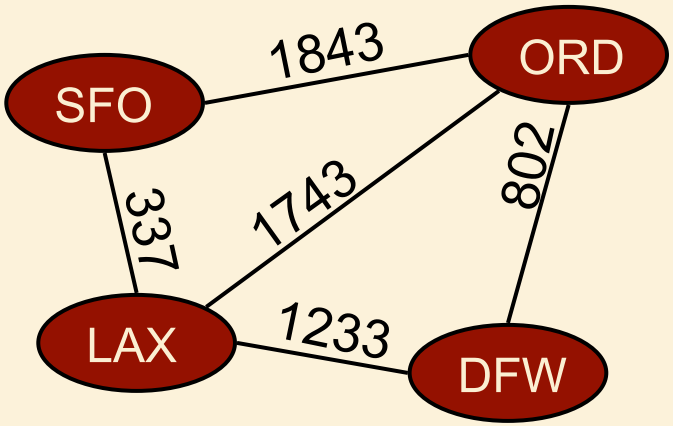
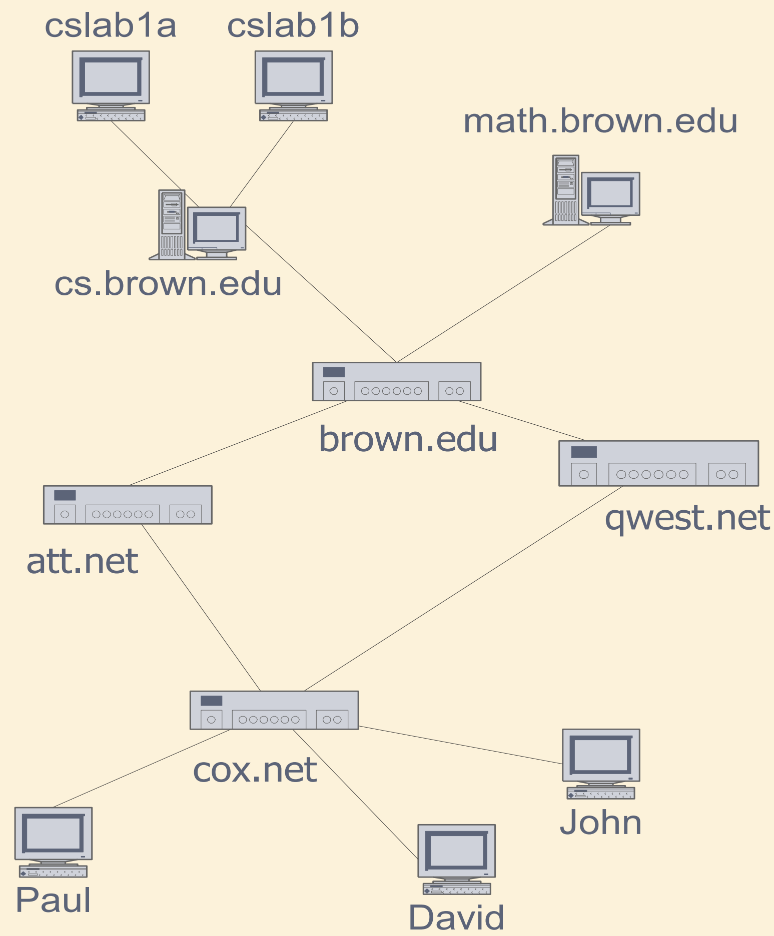
Graphs – ADTs and Implementations



Applications of Graphs

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



Outcomes

- By understanding this lecture, you should be able to:
  - Define basic terminology of graphs.
  - Use a graph ADT for appropriate applications.
  - Program standard implementations of the graph ADT.
  - Understand advantages and disadvantages of these implementations, in terms of space and run time.

Outline

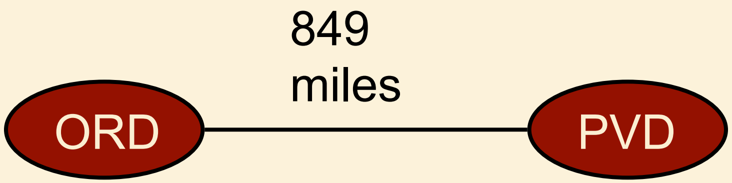
- Definitions
- Graph ADT
- Implementations

Outline

- **Definitions**
- Graph ADT
- Implementations

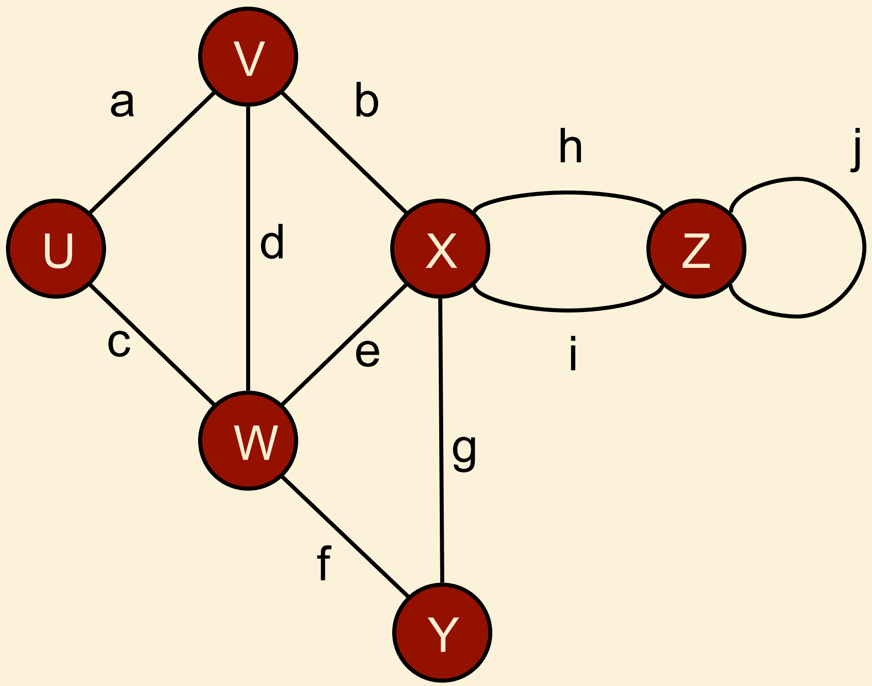
Edge Types

- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- Directed graph (Digraph)
  - all the edges are directed
  - e.g., route network
- Undirected graph
  - all the edges are undirected
  - e.g., flight network



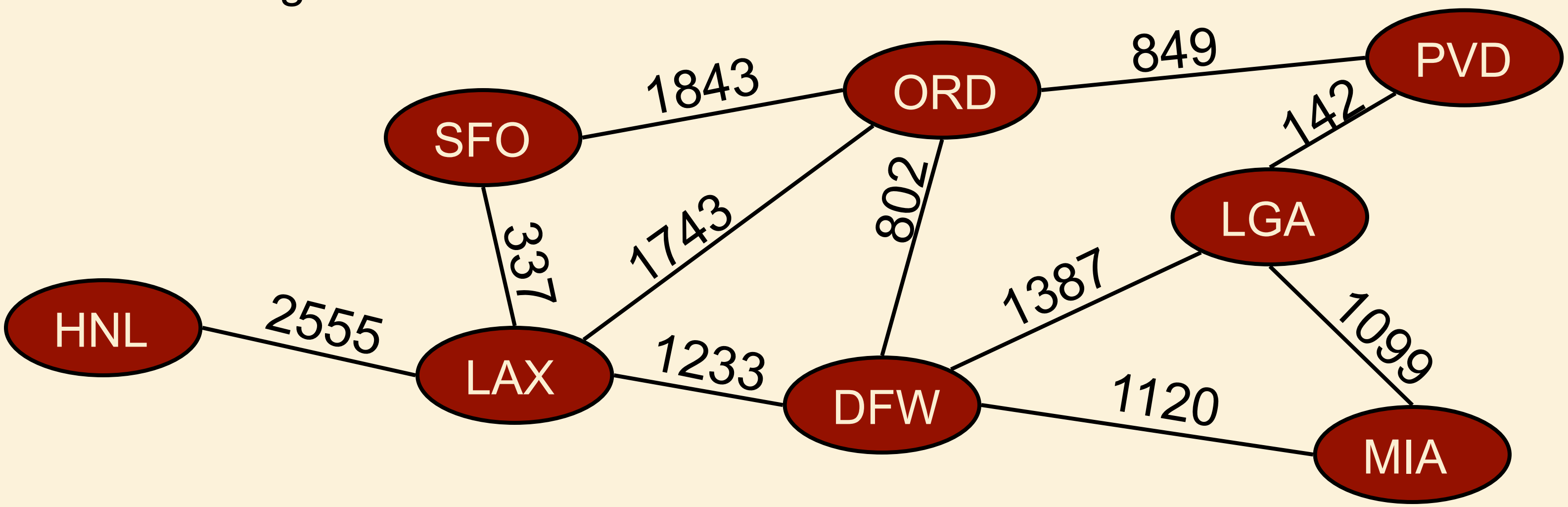
Vertices and Edges

- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop



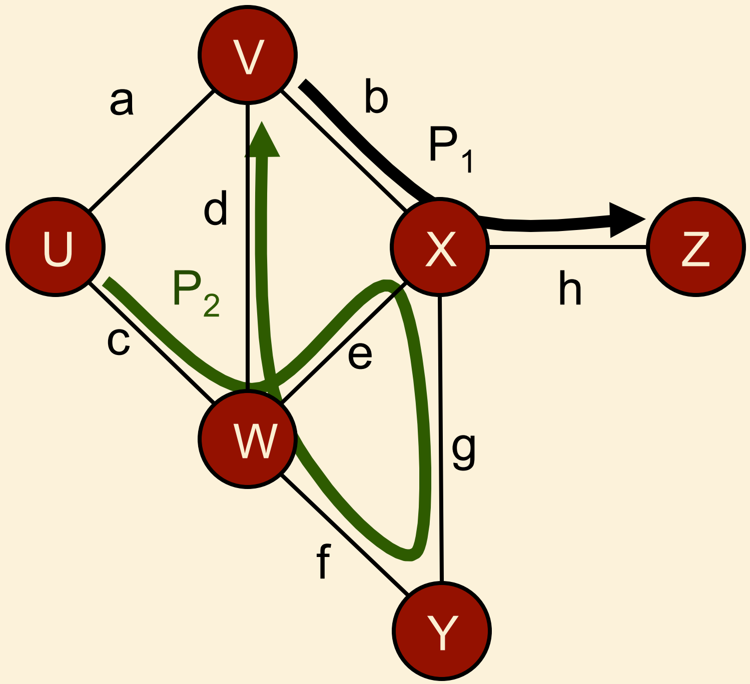
Graphs

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



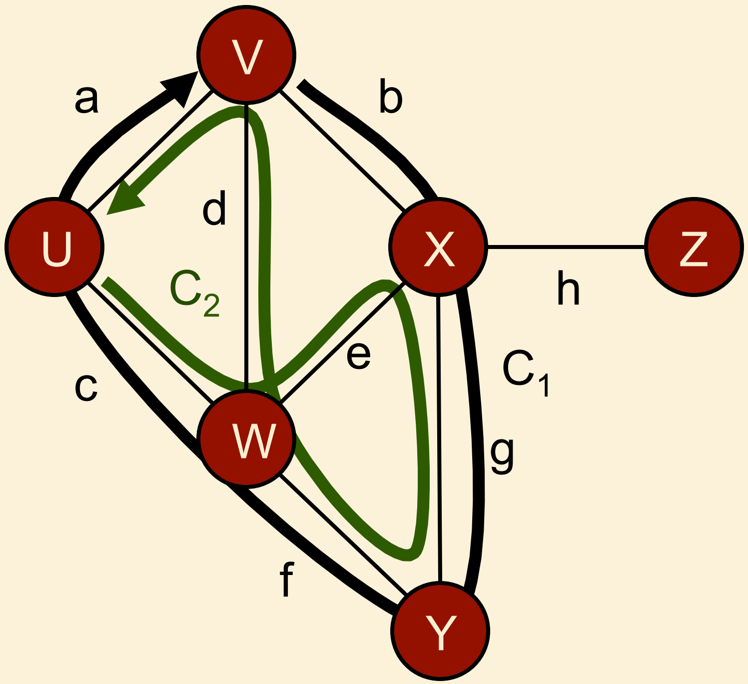
Paths

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



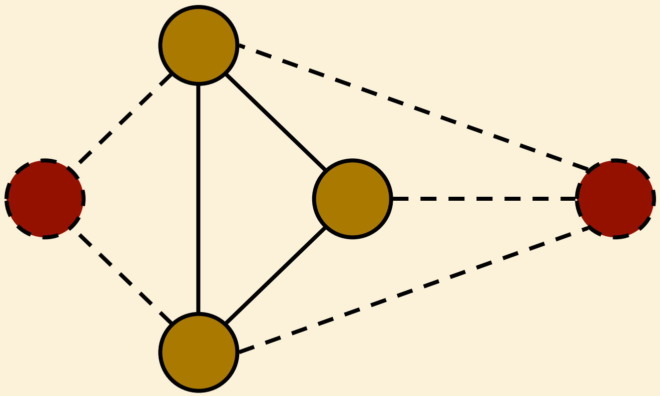
Cycles

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,V) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,U) is a cycle that is not simple

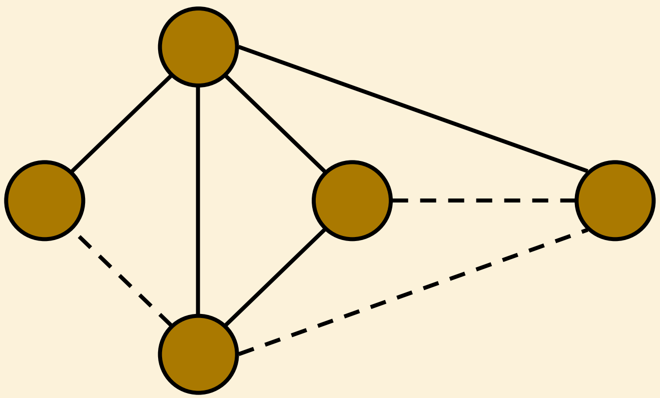


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



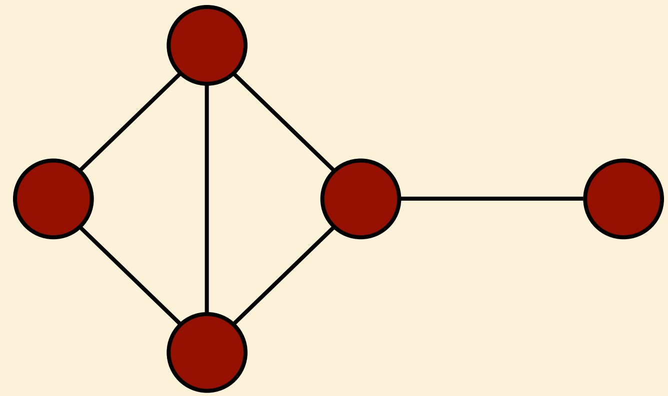
Subgraph



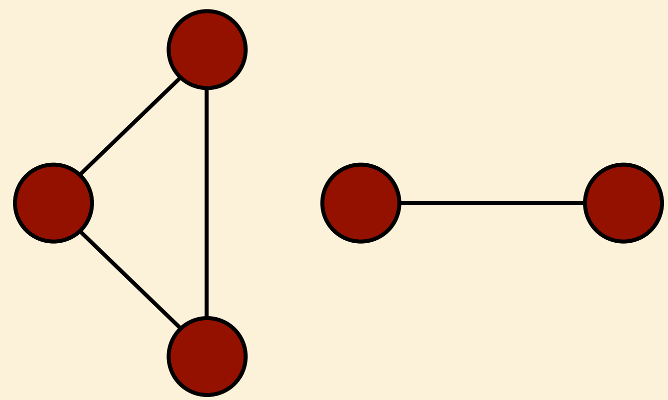
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G

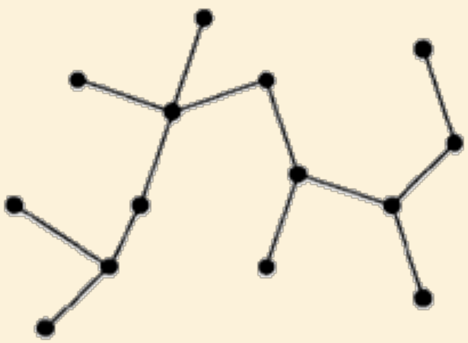


Connected graph

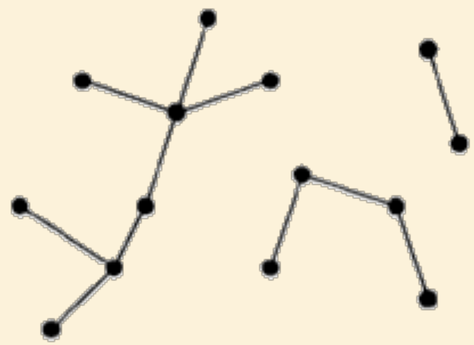


Non connected graph with two connected components

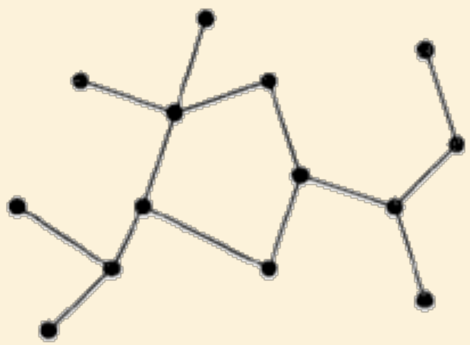
Trees



Tree



Forest



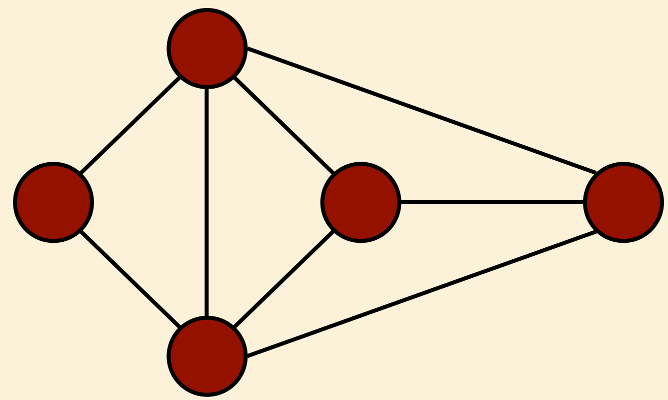
Graph with Cycle

A tree is a connected, acyclic, undirected graph.

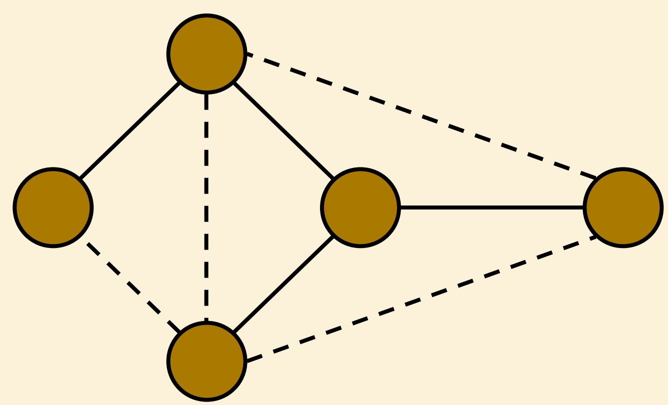
A forest is a set of trees (not necessarily connected)

Spanning Trees

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



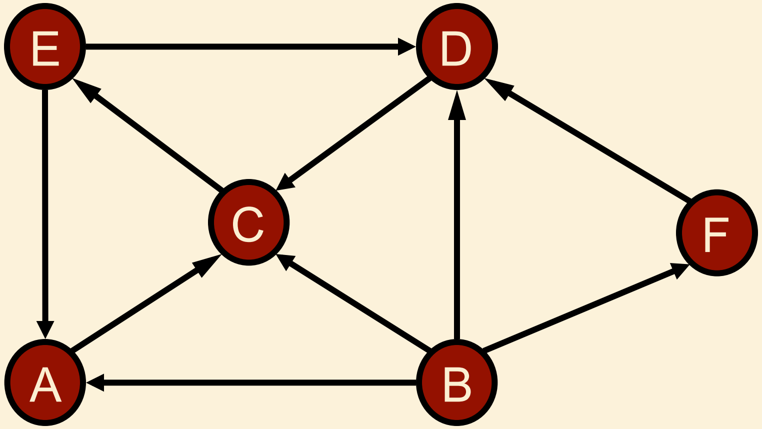
Graph



Spanning tree

Reachability in Directed Graphs

- A node w is ***reachable*** from v if there is a directed path originating at v and terminating at w.
  - E is reachable from B
  - B is not reachable from E



Properties

Property 1

$\Sigma_v \deg(v) = 2|E|$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no multiple edges

$|E| \leq |V|\,(|V| - 1)/2$

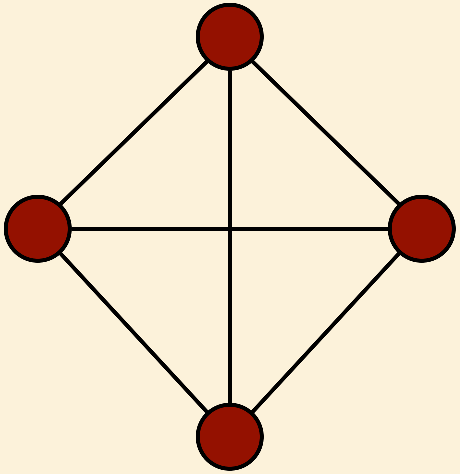
Proof: each vertex has degree at most $(|V| - 1)$

Q: What is the bound for a digraph?

$A: \ |E| \leq |V|(|V| - 1)$

Notation

$\|V\|$	number of vertices
$\|E\|$	number of edges
$\deg(v)$	degree of vertex v



Example

- $|V| = 4$
- $|E| = 6$
- $\deg(v) = 3$

Outline

- Definitions
- **Graph ADT**
- Implementations

Main Methods of the Graph ADT

- Accessor methods
  - numVertices(): Returns the number of vertices in the graph
  - numEdges(): Returns the number of vertices in the graph
  - getEdge(u, v): Returns edge from u to v
  - endVertices(e): an array of the two endvertices of e
  - opposite(v, e): the vertex opposite to v on e
  - outDegree(v): Returns number of outgoing edges
  - inDegree(v): Returns number of incoming edges

Main Methods of the Graph ADT

- Update methods
  - insertVertex(x): insert a vertex storing element x
  - insertEdge(u, v, x): insert an edge (u,v) storing element x
  - removeVertex(v): remove vertex v (and its incident edges)
  - removeEdge(e): remove edge e

Main Methods of the Graph ADT

- Iterator methods
  - incomingEdges(v): Incoming edges to v
  - outgoingEdges(v): Outgoing edges from v
  - vertices(): all vertices in the graph
  - edges(): all edges in the graph

Outline

- Definitions
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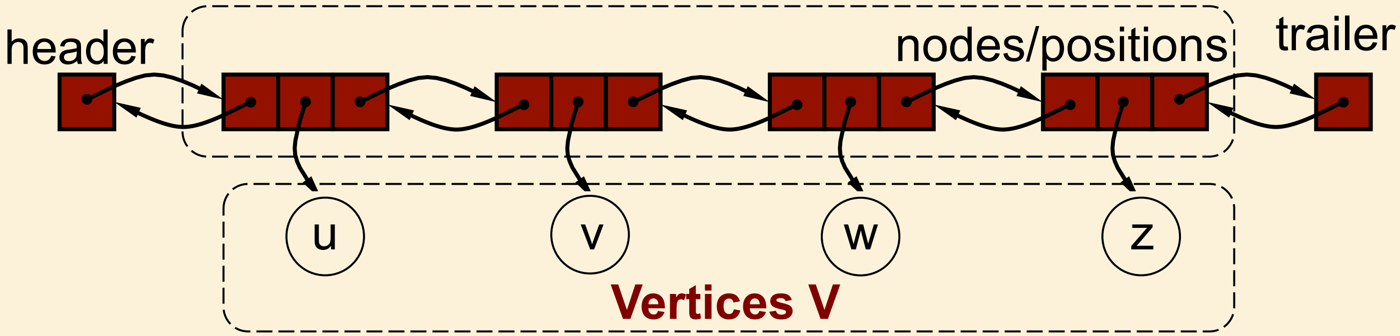
- **Implementations**

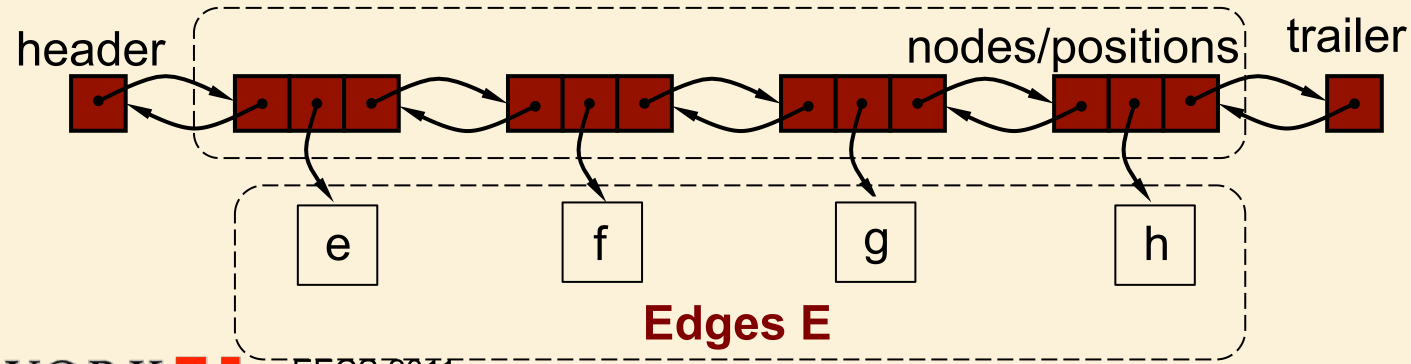
GTG Implementation (net.datastructures)

- There are many ways to implement the Graph ADT.
- We will follow the textbook implementation.

Vertex and Edge Lists

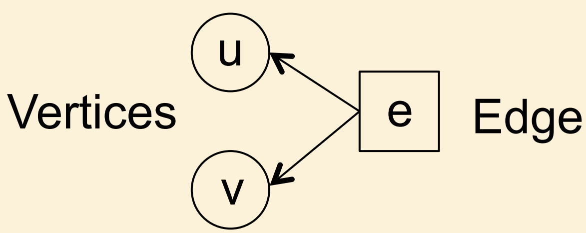
- A graph consists of a collection of vertices V and a collection of edges E.
- Each of these will be represented as a Positional List (Ch.7.3).
- In net.datastructures, Positional Lists are implemented as doubly-linked lists.

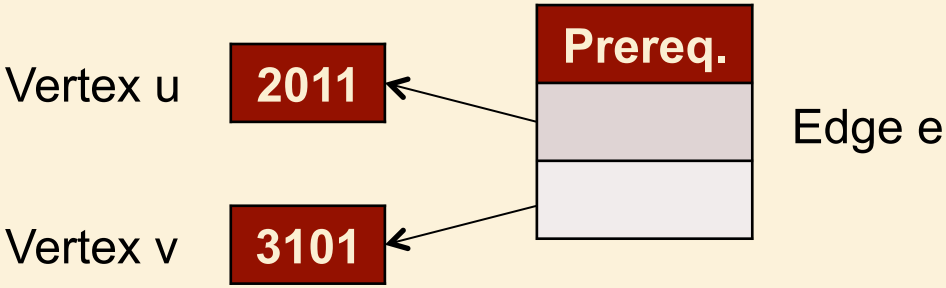




Vertices and Edges

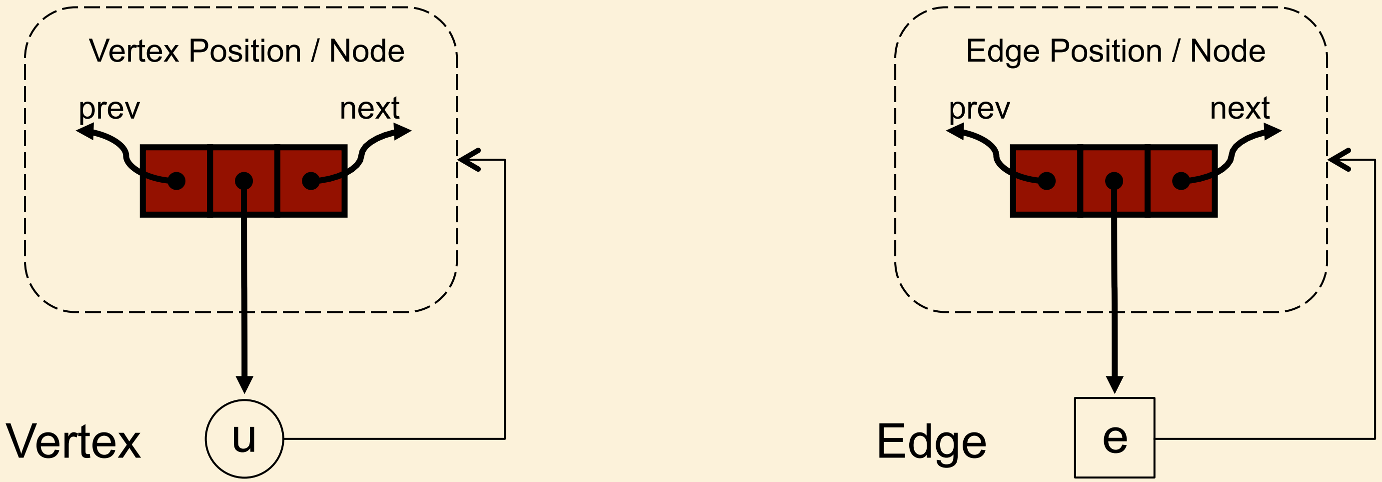
- Each vertex v stores an element containing information about the vertex.
  - For example, if the graph represents course dependencies, the vertex element might store the course number.
- Each edge e stores an element containing information about the edge.
  - e.g., pre-requisite, co-requisite.
- In addition, each edge must store references to the vertices it connects.





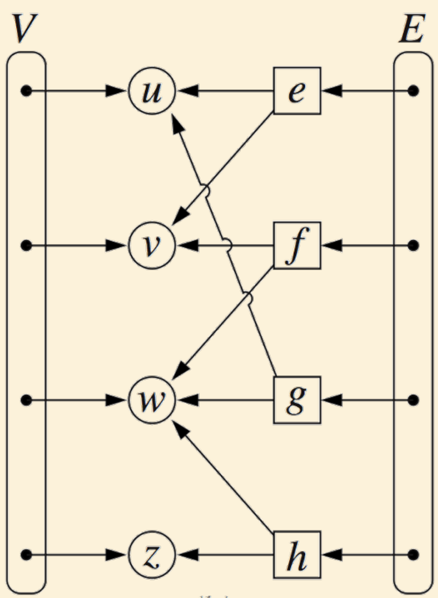
Vertices and Edges

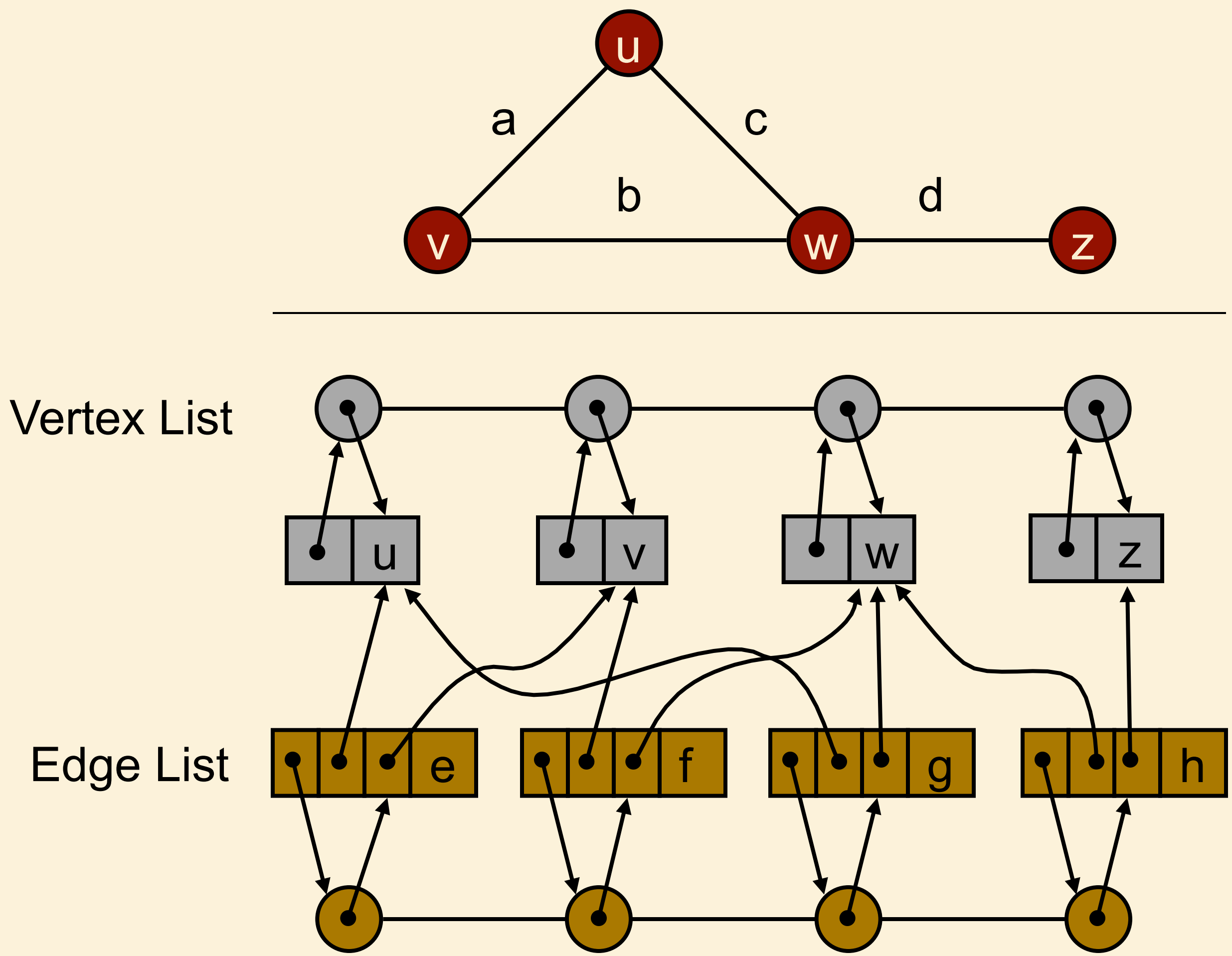
- To facilitate efficient removal of vertices and edges, we will make both location aware:
  - A reference to the Position in the Positional List will be stored in the element.



Edge List Implementation

➢ This organization yields an Edge List Structure





Performance of Edge List Implementation

- Edge List implementation does not provide efficient access to edge information from vertex list.

▪ n vertices, m edges ▪ no parallel edges ▪ no self-loops	Edge List
Space	$n + m$
incomingEdges(v) outgoingEdges(v)	m
getEdge(u, v)	m
insertVertex(x)	1
insertEdge(u, v, x)	1
removeVertex(v)	m
removeEdge(e)	1

Other Graph Implementations

- Can we come up with a graph implementation that improves the efficiency of these basic operations?
  - Adjacency List
  - Adjacency Map
  - Adjacency Matrix

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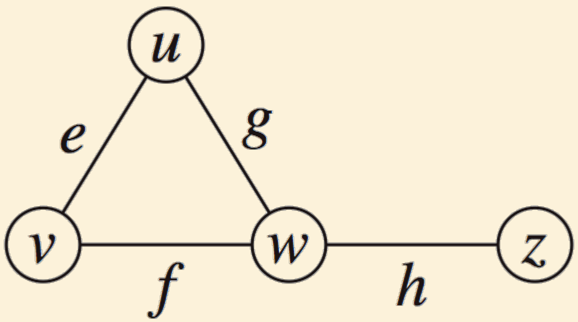
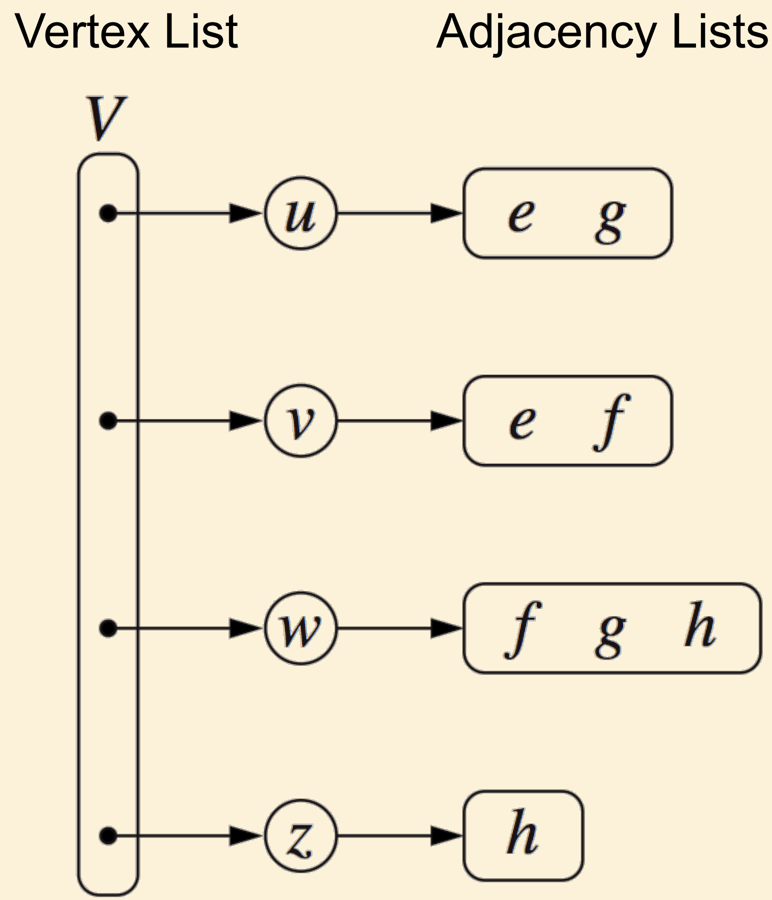
- Can we come up with a graph implementation that improves the efficiency of these basic operations?
  - **Adjacency List**
  - Adjacency Map
  - Adjacency Matrix

Adjacency List Implementation

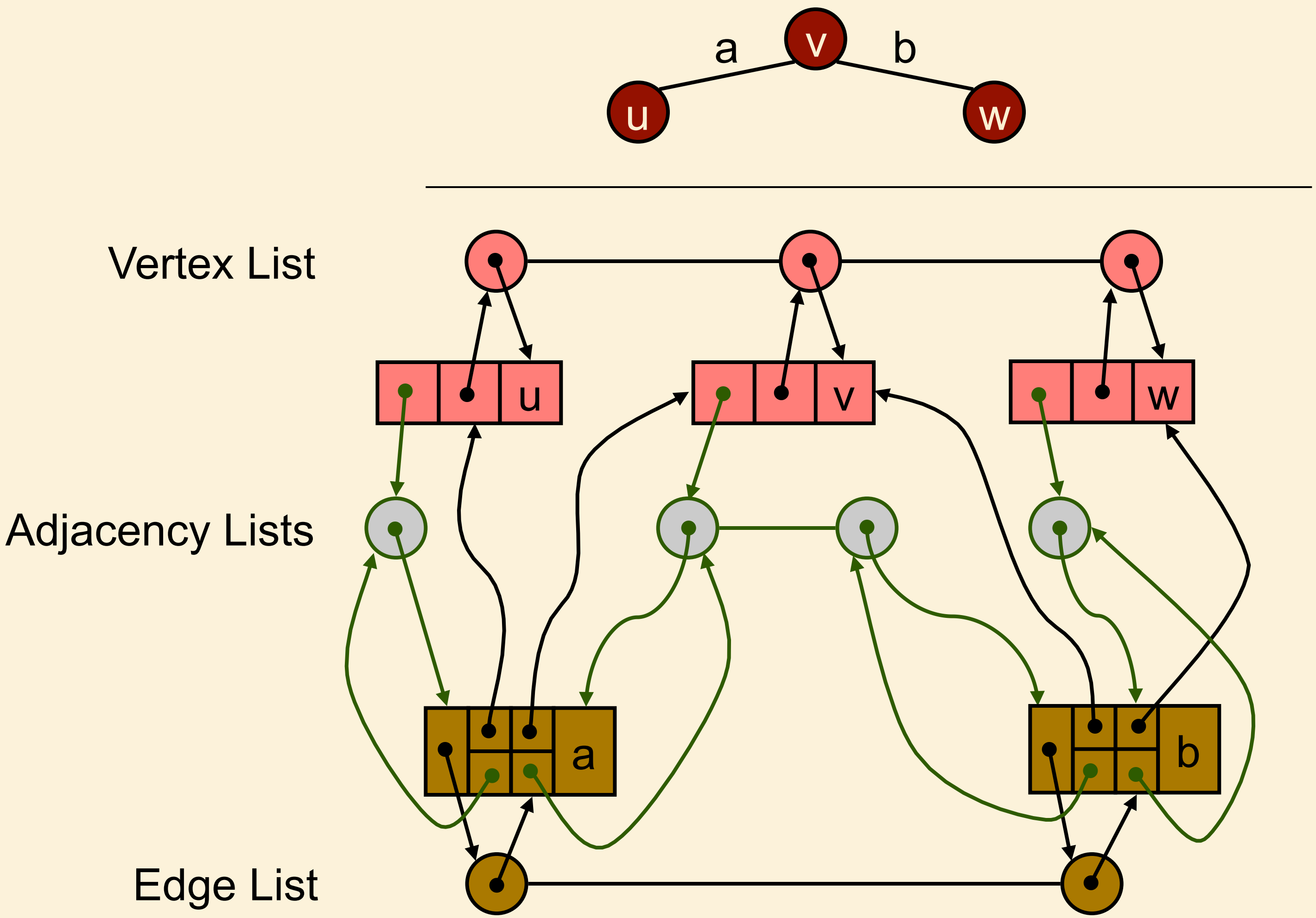
- An Adjacency List implementation augments each vertex element with Positional Lists of incoming and outgoing edges.





Adjacency List Implementation

- An Adjacency List implementation augments each vertex element with lists of incoming and outgoing edges.



Performance of Adjacency List Implementation

- Adjacency List implementation improves efficiency without increasing space requirements.

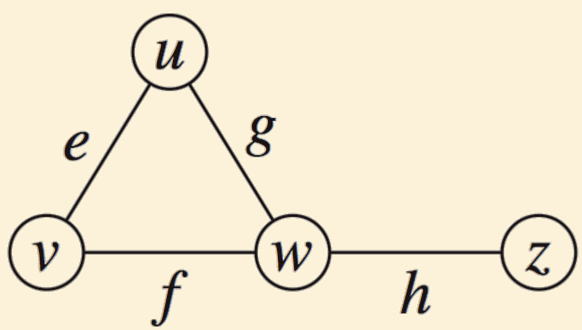
▪ n vertices, m edges ▪ no parallel edges ▪ no self-loops	Edge List	Adjacency List
Space	$n + m$	$n + m$
incomingEdges(v) outgoingEdges(v)	m	$\deg(v)$
getEdge(u, v)	m	$\min(\deg(u), \deg(v))$
insertVertex(x)	1	1
insertEdge(u, v, x)	1	1
removeVertex(v)	m	$\deg(v)$
removeEdge(e)	1	1

Other Graph Implementations

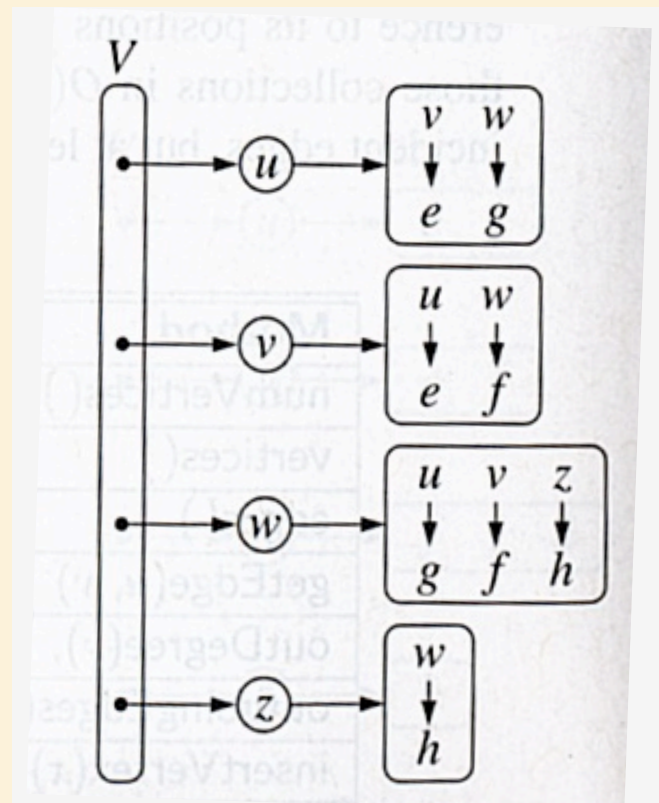
- Can we come up with a graph implementation that improves the efficiency of these basic operations?
  - Adjacency List
  - **Adjacency Map**
  - Adjacency Matrix

Adjacency Map Implementation

- An Adjacency Map implementation augments each vertex element with an Adjacency Map of edges
  - Each entry consists of:
    - Key = opposite vertex
    - Value = edge
  - Implemented as a hash table.



Vertex List Adjacency Maps



Performance of Adjacency Map Implementation

➢ Adjacency Map implementation improves expected run time of getEdge(u,v):

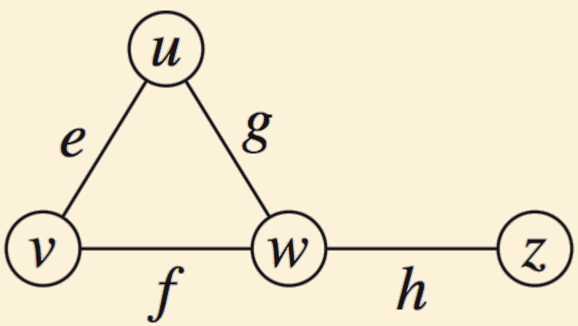
▪ n vertices, m edges ▪ no parallel edges ▪ no self-loops	Edge List	Adjacency List	Adjacency Map
Space	$n + m$	$n + m$	$n + m$
incomingEdges(v), outgoingEdges(v)	m	deg(v)	deg(v)
getEdge(u, v)	m	min(deg(u), deg(v))	1 (exp.)
insertVertex(x)	1	1	1
insertEdge(u, v, x)	1	1	1 (exp.)
removeVertex(v)	m	deg(v)	deg(v)
removeEdge(e)	1	1	1 (exp.)

Other Graph Implementations

- Can we come up with a graph implementation that improves the efficiency of these basic operations?
  - Adjacency List
  - Adjacency Map
  - **Adjacency Matrix**

Adjacency Matrix Implementation

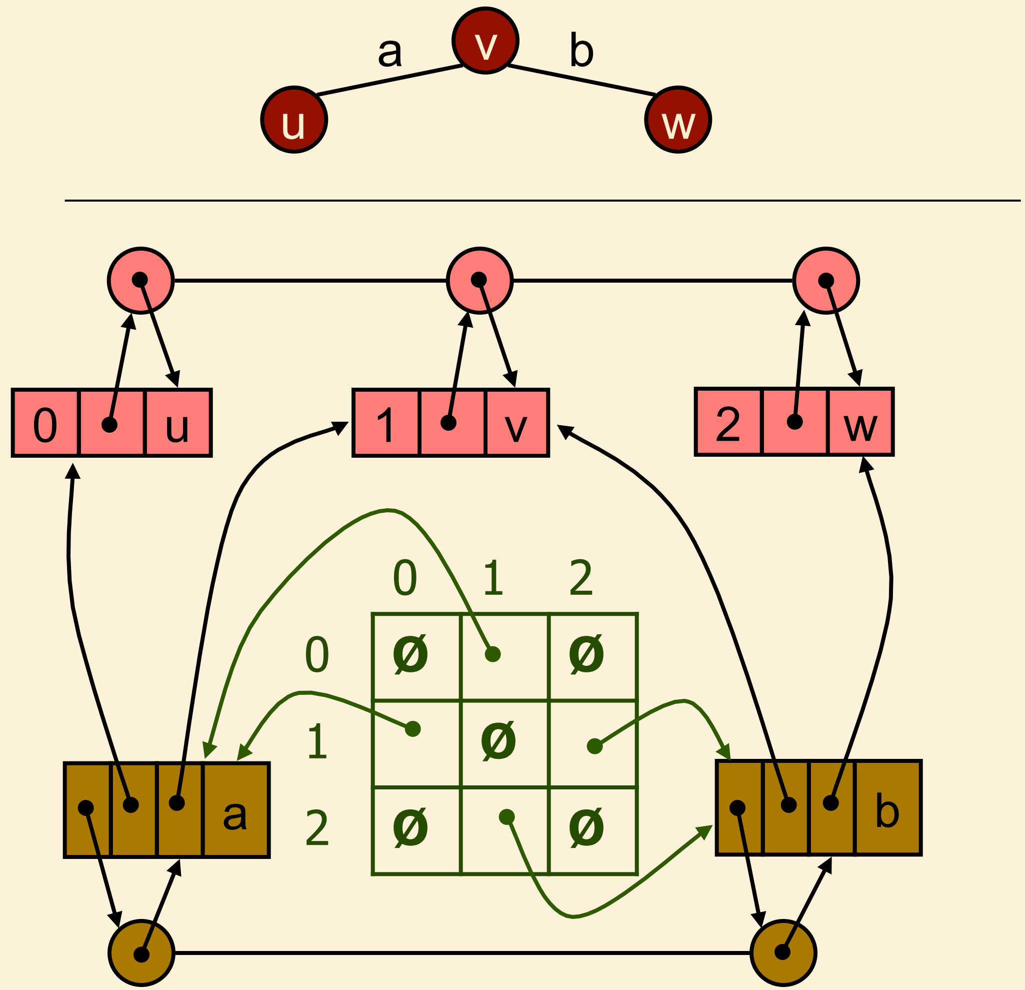
- In an Adjacency Matrix implementation we map each of the n vertices to an integer index from [0…n-1].
- Then a 2D n x n array A is maintained:
  - If edge (i, j) exists, A[i, j] stores a reference to the edge.
  - If edge (i, j) does not exist, A[i, j] is set to null.



Vertex List | Adjacency Matrix

		0	1	2	3
$u \rightarrow$	0		e	g	
$v \rightarrow$	1	e		f	
$w \rightarrow$	2	g	f		h
$z \rightarrow$	3			h	

Adjacency Matrix Structure



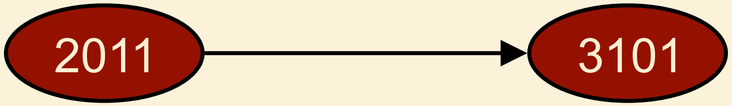
Performance of Adjacency Matrix Implementation

- Requires more space.
- Slow to get incoming / outgoing edges
- Very slow to insert or remove a vertex (array must be resized)

▪ n vertices, m edges ▪ no parallel edges ▪ no self-loops	Edge List	Adjacency List	Adjacency Map	Adjacency Matrix
Space	$n + m$	$n + m$	$n + m$	n^2
incomingEdges(v), outgoingEdges(v)	m	$\deg(v)$	$\deg(v)$	n
getEdge(u, v)	m	$\min(\deg(u), \deg(v))$	1 (exp.)	1
insertVertex(x)	1	1	1	n^2
insertEdge(u, v, x)	1	1	1 (exp.)	1
removeVertex(v)	m	$\deg(v)$	$\deg(v)$	n^2
removeEdge(e)	1	1	1 (exp.)	1

A4Q2: Course Prerequisites

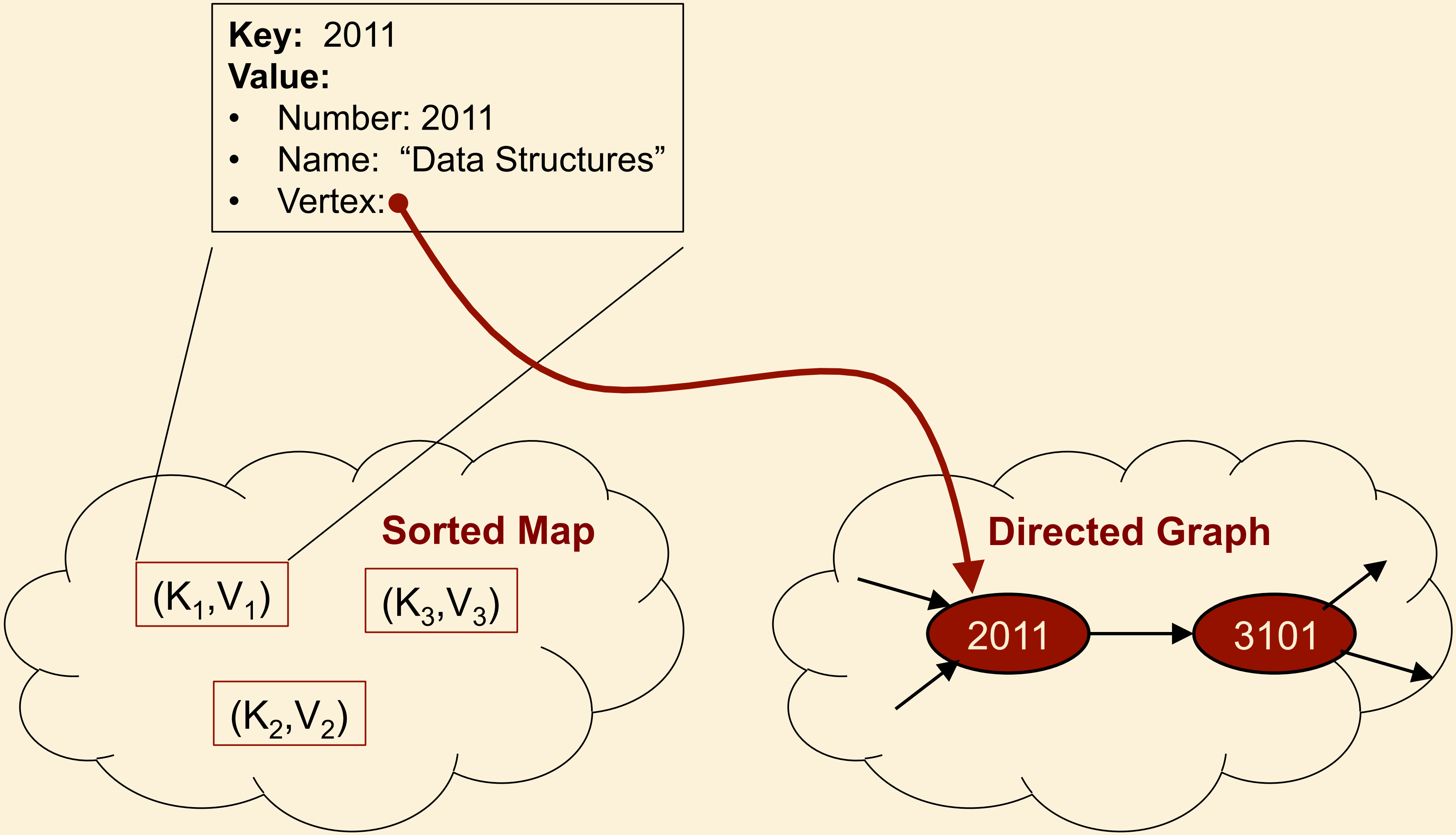
- In most post-secondary programs, courses have prerequisites.
- For example, you cannot take EECS 3101 until you have passed EECS 2011.
- How can we represent such a system of dependencies?
- A natural choice is a **directed graph**.
  - Each vertex represents a course
  - Each directed edge represents a prerequisite
    - A directed edge from Course U to Course V means that Course U must be taken before Course V.



A4Q2: Course Prerequisites

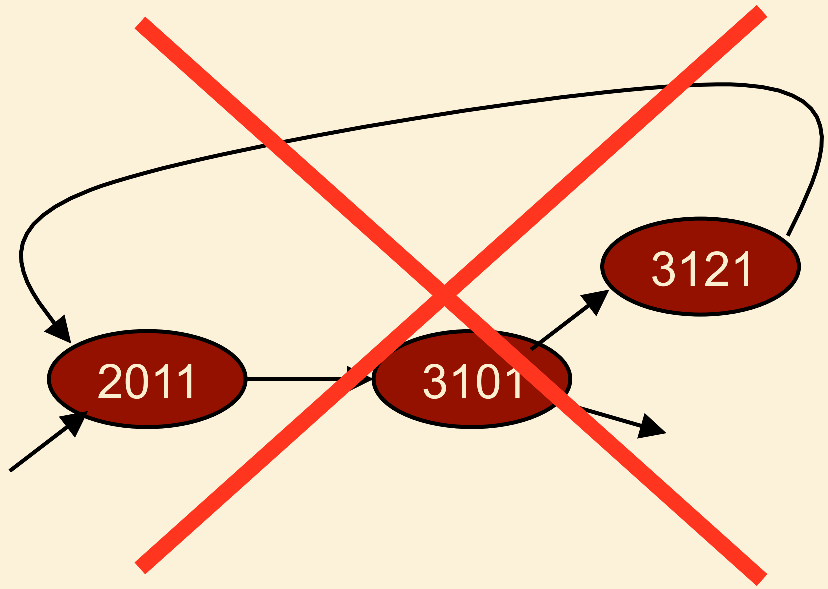
- We also want to be able to find the information for a particular course quickly.
- The course number provides a convenient key that can be used to organize course records in a sorted map, implemented as a binary search tree (cf. A3Q1).
- Thus it makes sense to represent courses using both a sorted map (for efficient access) and a directed graph (to represent dependencies).
- By storing a reference to the directed graph vertex for a course in the sorted map, we can efficiently access course dependencies.

A4Q2: Course Prerequisites



A4Q2: Course Prerequisites

- It is important that the course prerequisite graph be a directed acyclic graph (DAG). Why?

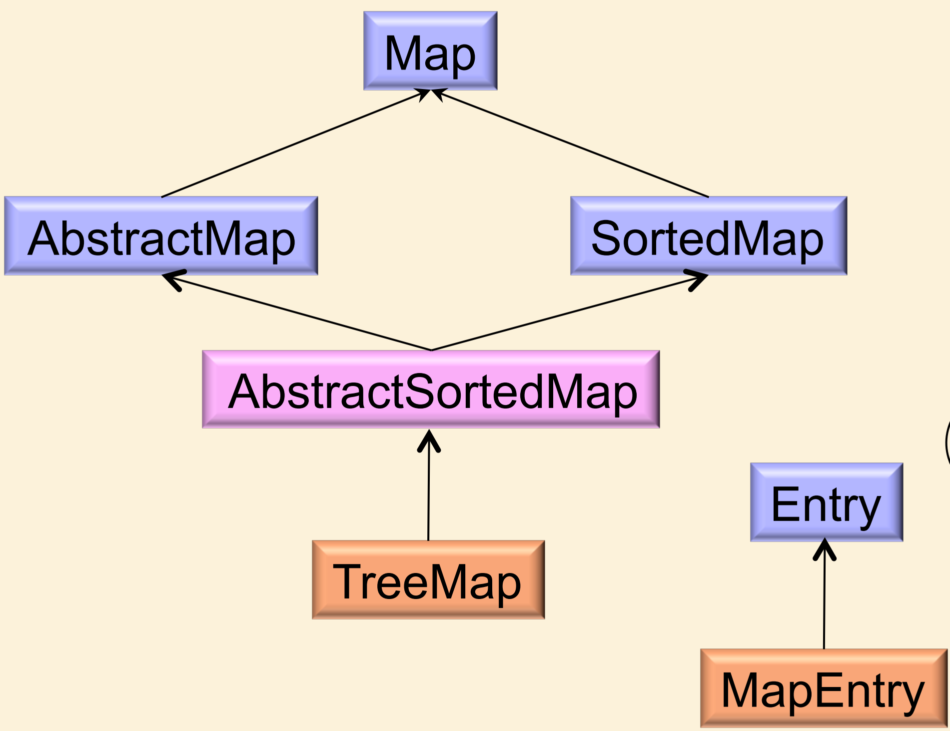


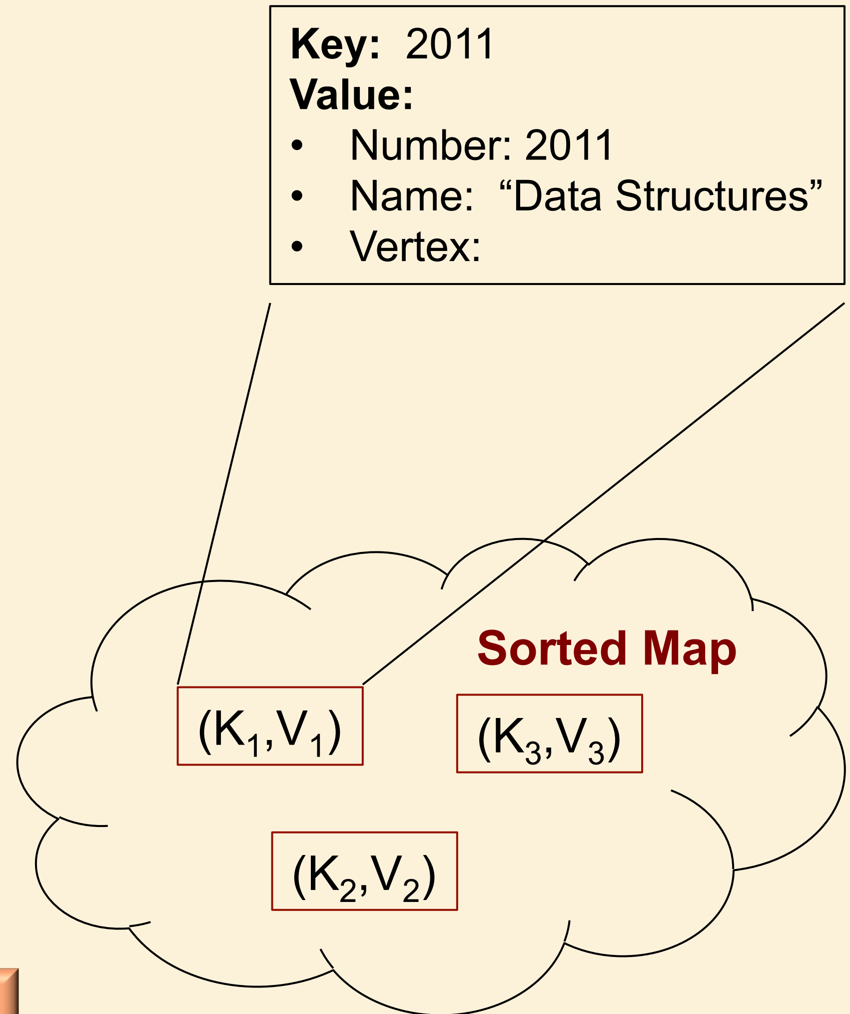
A4Q2: Course Prerequisites

- In this question, you are provided with a basic implementation of a system to represent courses and dependencies.
- Methods for adding courses and getting prerequisites are provided.
- You need only write the method for adding a prerequisite.
- This method will use a depth-first-search algorithm (also provided) that can be used to prevent the addition of prerequisites that introduce cycles.

A4Q2: Implementation using net.datastructures

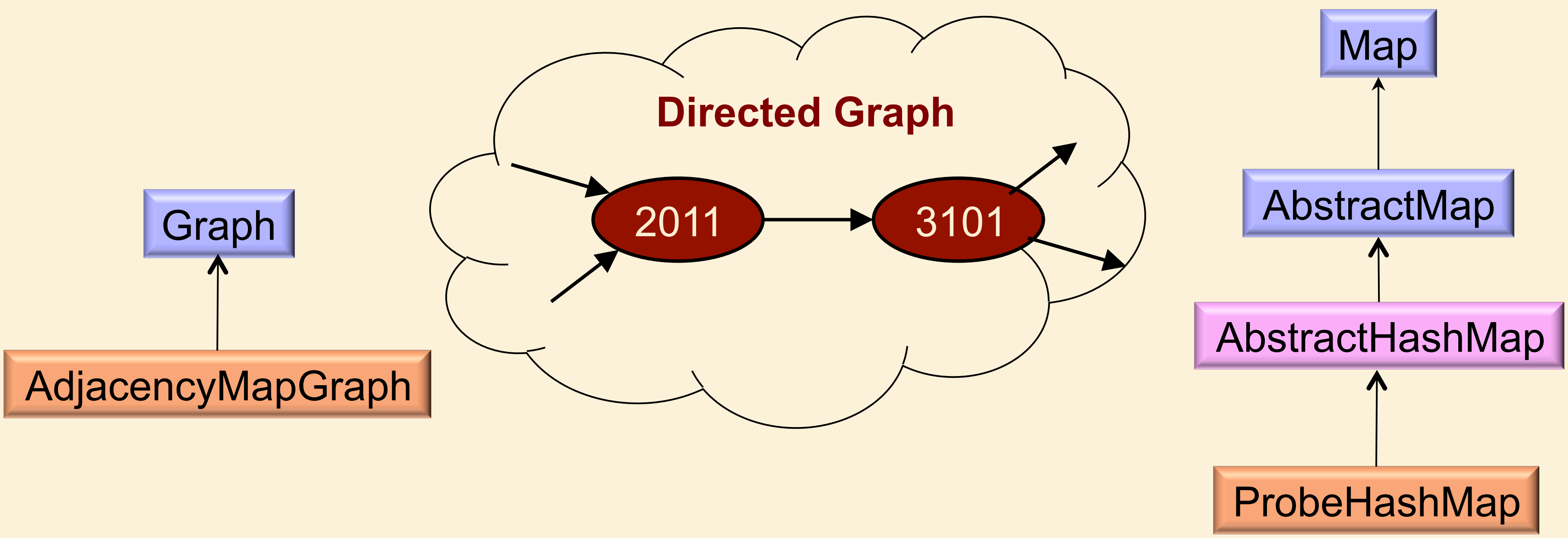
- We use the TreeMap class to represent the sorted map (cf. A3Q1).





A4Q2: Implementation using net.datastructures

- We use the **AdjacencyMapGraph** class to represent the directed graph.
- This implementation uses **ProbeHashMap**, a linear probe hash table, to represent the incoming and outgoing edges for each vertex.



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- Implementations

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  - Use a graph ADT for appropriate applications.
  - Program standard implementations of the graph ADT.
  - Understand advantages and disadvantages of these implementations, in terms of space and run time.