MATH/EECS 1028: Discrete Math for Engineers
Winter 2015
Tutorial 7 (Week of Mar 2, 2015)

Notes:

1. Assume $\mathbb{R}$ to denote the real numbers, $\mathbb{Z}$ to denote the set of integers $(\ldots, -2, -1, 0, 1, 2, \ldots)$ and $\mathbb{N}$ to denote the natural numbers $(1, 2, 3, \ldots)$.
2. Topics: Cardinality, Induction, Pigeonhole principle.
3. Note to the TA: Attendance will be taken this week on Monday. The Friday section will have a quiz this week.

Questions:

1. Prove that among any given $n+1$ positive integers, there are always two whose difference is divisible by n.

   Hint: Use the Pigeonhole Principle.
2. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x}{1-x}$. Define $f^2(x) = f(f(x)), f^3(x) = f(f(f(x)))$ and so on. Guess the form for $f^n(x)$ and prove your answer correct using induction on n.
3. Show that if 7 integers are selected from the first 10 positive integers (i.e., the numbers 1 thorugh 10), there must be at least 2 pairs of these integers with sum 11.
4. Let k be a positive integer and x be real. Prove using induction that if $x + \frac{1}{x}$ is an integer then $x^k + \frac{1}{x^k}$ is also an integer.
5. Consider the set of all fractions of the form $\frac{n}{n+\sqrt{n}}$, where $n \in \mathbb{Z}, n > 0$. Is the set countable? Prove your answer.
6. Prove using mathematical induction that if n non-parallel straight lines on the plane intersect at a common point, they divide the plane into $2n$ regions.
7. Use strong induction to show that every positive integer can be written as a sum of distinct powers of 2, that is, as a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$ and so on.

   Hint: For the inductive step, separately consider the case where $k+1$ is even and where it is odd. Where it is even, note that $\frac{k+1}{2}$ is an integer.
8. The digital sum of a number is defined as the sum of its decimal digits. For example, the digital sum of 386 is $3 + 8 + 6 = 17$. Suppose 35 two-digit numbers are selected. Prove that there are three of them with the same digital sum.
9. Given any n natural numbers, the sum of some non-empty subset of them is divisible by n.