MATH/EECS 1028: Discrete Math for Engineers
Winter 2015
Tutorial 7 (Week of Mar 2, 2015)

Notes:

1. Assume $\mathbb{R}$ to denote the real numbers, $\mathbb{Z}$ to denote the set of integers $(\ldots, -2, -1, 0, 1, 2, \ldots)$ and $\mathbb{N}$ to denote the natural numbers $(1, 2, 3, \ldots)$.
2. Topics: Cardinality, Induction, Pigeonhole principle.
3. Note to the TA: Attendance will be taken this week on Monday. The Friday section will have a quiz this week.

Questions:

1. Prove that among any given $n+1$ positive integers, there are always two whose difference is divisible by $n$.

   Hint: Use the Pigeonhole Principle.

   **Solution:**

   Define $j$ modulo $n$ to be the remainder obtained when $j$ is divided by $n$. Then consider each of the $n+1$ integers modulo $n$. Thus there are $n+1$ numbers and only $n$ possible values, since valid remainders upon division by $n$ are $0, 1, \ldots, n-1$. By the Pigeonhole Principle there must exist two numbers $a, b$ that have the same value $r$ modulo $n$, i.e., $a = pn + r$ and $b = qn + r$ for some integers $p, q$. Therefore $a - b = (p - q)n$ is a multiple of $n$.

2. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x}{1-x}$. Define $f^2(x) = f(f(x)), f^3(x) = f(f(f(x)))$ and so on. Guess the form for $f^n(x)$ and prove your answer correct using induction on $n$.

   **Solution:** We can check that

$$\begin{aligned} f(f(x)) &= \frac{f(x)}{1-f(x)} \\ &= \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} \\ &= \frac{x}{1-2x} \end{aligned}$$

   We can continue and check that $f(f(f(x))) = \frac{x}{1-3x}$ and so on and can conjecture that $f^n(x) = \frac{x}{1-nx}$. Let us prove this using induction on $n$.

   Base Case: For $n = 1$ the definition of $f(x)$ satisfies the hypothesis.

Inductive Step: Suppose the statement is true for $n = k$ so that $f^k(x) = \frac{x}{1-kx}$. We need to prove that the statement holds for $n = k+1$, i.e., $f^{k+1}(x) = \frac{x}{1-(k+1)x}$.

$$\begin{aligned} f^{k+1}(x) &= \frac{f^k(x)}{1-f^k(x)} \\ &= \frac{\frac{x}{1-kx}}{1-\frac{x}{1-kx}} \\ &= \frac{\frac{x}{1-kx}}{\frac{1-(k+1)x}{1-kx}} \\ &= \frac{x}{1-(k+1)x} \end{aligned}$$

Thus the conjecture is true.

3. Show that if 7 integers are selected from the first 10 positive integers (i.e., the numbers 1 thorugh 10), there must be at least 2 pairs of these integers with sum 11.

   **Solution:** Note that there are 5 pairs that sum to 11" $1+10, \ldots, 5+6$. Think of these pairs as bins and the 7 integers being chosen as the balls. Note also that each bin can have at most 2 balls (selections). So if there are 7 numbers, at least two bins will have two balls each. So there are at least 2 pairs of integers that sum to 11.

4. Let $k$ be a positive integer and $x$ be real. Prove using induction that if $x + \frac{1}{x}$ is an integer then $x^k + \frac{1}{x^k}$ is also an integer.

   **Solution:**

   We prove this by using strong induction on $k$.

   Base Case: $k = 1$. True, since $x + \frac{1}{x}$ is given to be an integer.

   Inductive step: Assume the statement is true for $k = m$ and $k = m-1$. So $x^{m-1} + \frac{1}{x^{m-1}}$ and $x^m + \frac{1}{x^m}$ are integers. Then for $k = m+1$,

   $$x^{m+1} + \frac{1}{x^{m+1}} = \left(x + \frac{1}{x}\right)\left(x^m + \frac{1}{x^m}\right) - \left(x^{m-1} + \frac{1}{x^{m-1}}\right)$$

   Since each of the three terms on the right hand side are integers, the left hand side is an integer. Hence by the principle of mathematical induction, the given statement is true.

5. Consider the set of all fractions of the form $\frac{n}{n+\sqrt{n}}$, where $n \in \mathbb{Z}, n > 0$. Is the set countable? Prove your answer.

   **Solution:** This set is countable because we can define a bijection $f : \mathbb{N} \to \mathbb{R}, f(n) = \frac{n}{n+\sqrt{n}}$.

   Note that $f(n) = \frac{1}{1+1/\sqrt{n}}$. This mapping is one-to-one because if $f(n_1) = f(n_2)$ for some $n_1, n_2$ then $\frac{1}{1+1/\sqrt{n_1}} = \frac{1}{1+1/\sqrt{n_2}}$ which implies $n_1 = n_2$. The mapping $f$ is onto because every element in the given set is indexed by a natural number $n$. Any set that has a bijection from $\mathbb{N}$ is countable.

6. Prove using mathematical induction that if $n$ non-parallel straight lines on the plane intersect at a common point, they divide the plane into $2n$ regions.

   **Solution:**

   We prove this by using induction on $n$.

   Base Case: $n = 1$. True, since one straight line divides the plane into 2 regions.

   Inductive step: Suppose that the hypothesis is true for $n = m$, so that any $m$ non-parallel straight lines on the plane intersecting at a common point divide the plane into $2m$ regions. Then for $n = m + 1$, we have to show that any $m + 1$ non-parallel straight lines on the plane intersecting at a common point divide the plane into $2m+2$ regions.

   Choose any set of $m$ lines from a given set of $m + 1$ non-parallel straight lines on the plane intersecting at a common point. By the inductive hypothesis, these lines divide the plane into $2m$ regions. Since the $m + 1^{th}$ line also passes through the common intersection, it passes through exactly two of the $2m$ regions. Since it cuts each of these two regions in two parts, it creates a total of $2m + 2$ regions.

   Hence by the principle of mathematical induction, the given statement is true.

7. Use strong induction to show that every positive integer can be written as a sum of distinct powers of 2, that is, as a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$ and so on.
   Hint: For the inductive step, separately consider the case where $k+1$ is even and where it is odd. Where it is even, note that $\frac{k+1}{2}$ is an integer.

   **Solution:** Note: The word distinct is crucial, otherwise every number can be written as a sum of 1's.

   For the base case we use $n = 1$. Then the statement is true because $1 = 2^0$.

   For the inductive step let the statement be true for all $n \leq k$. Then for $n = k + 1$ we consider 2 cases.

   Case 1: $k + 1$ is even. Then $k + 1 = 2p$ for some integer $p > 0$, and by the inductive hypothesis $p$ can be written as a sum of distinct powers of 2. This implies that $2p$ is a sum of distinct powers of 2, because each distinct power of 2 in $p$ is multiplied by 2.

   Case 2: $k + 1$ is odd. So $k$ is even. By the inductive hypothesis, $k$ is a sum of distinct powers of 2 and further it cannot have $2^0$ because every other power of 2 is even and thus the sum $k$ would be odd. It follows that $k + 1$ can be expressed as a sum of distinct powers of 2 by adding $2^0 = 1$ to the sum of distinct powers of 2 that form $k$.

   Thus in each case $k + 1$ can be expressed as a sum of distinct powers of 2.

8. The digital sum of a number is defined as the sum of its decimal digits. For example, the digital sum of 386 is $3 + 8 + 6 = 17$. Suppose 35 two-digit numbers are selected. Prove that there are three of them with the same digital sum.

   **Solution:** This is a little tricky. The possible values of the digital sum of 2 digit numbers range from 1 to 18. So the pigeonhole principle does not give you the answer. You have to look closely and see that the sums 1 and 18 only correspond to one number each (10, 99, respectively). If these are not chosen, we have 33 numbers with 16 possible sums. If only 1 of these is chosen we have 34 other numbers with 16 other sums. If both are chosen we have 33 other numbers from 16 possible sums. In all these cases the Pigeonhole Principle gives you the answer.

9. Given any $n$ natural numbers, the sum of some non-empty subset of them is divisible by $n$.

   **Solution:** This is a hard problem. In these problems the key is to set up the problem so that the Pigeonhole Principle can be applied.

   Suppose the numbers are $a_1, a_2, \ldots, a_n$. Then consider the partial sums $s_1 = a_1, s_2 = a_1 + a_2, s_3 = a_1 + a_2 + a_3, \ldots, s_n = a_1 + a_2 + \ldots + a_n$.

   Consider the $s_i$'s modulo n. Either one of them is zero, and we are done, or 2 of them have the same non-zero value. Call these $s_i, s_j, i < j$. Then $s_j - s_i$ must be divisible by $n$. Note that $s_j - s_i = s_{i+1} + \ldots + s_j$. So we are done in this case as well.