MATH/EECS 1028: Discrete Math for Engineers
Winter 2015
Tutorial 5 (Week of Feb 9, 2015)

Notes:

1. Assume $\mathbb{R}$ to denote the real numbers, $\mathbb{Z}$ to denote the set of integers $(\ldots, -2, -1, 0, 1, 2, \ldots)$ and $\mathbb{N}$ to denote the natural numbers $(1, 2, 3, \ldots)$.
2. Topics: Sequences, Logic.
3. Note to the TA: Attendance will be taken this week on Friday. Monday sections have a quiz this week.

Questions:

1. Predicates.

   Translate the following into English where $R(x)$ is "x is a comedian" and $H(x)$ is "x hops" and the domain consists of all animals. Then write down the negation of each statement in logic.

   (a) $\forall x(R(x) \to H(x))$

   Solution: "All comedians hop".
   The negation is

$$\begin{aligned} \neg(\forall x(R(x) \to H(x))) &\equiv \exists x \neg(R(x) \to H(x)) \\ &\equiv \exists x \neg(\neg R(x) \vee H(x)) \\ &\equiv \exists x(R(x) \wedge \neg H(x)) \end{aligned}$$

   (b) $\exists x(R(x) \wedge H(x))$

   Solution: "There is at least one comedian who hops".
   The negation is

$$\neg(\exists x(R(x) \wedge H(x))) \equiv \forall x \neg(R(x) \wedge H(x)) \quad (0.1)$$
$$\equiv \forall x(\neg R(x) \vee \neg H(x)) \quad (0.2)$$

2. Express using logical operators, quantifiers and predicates: "The negation of a contradiction is a tautology".

   Solution from the text: Let $T(x)$ mean that x is a tautulogy and $C(x)$ mean that x is a contradiction. Then $\forall x(C(x) \to T(\neg x)$.

   Note: The solution should also mention that the domain is all propositions. Since x is a proposition, $\neg x$ is well defined.

3. Let $P(x), Q(x), R(x), S(x)$ be the statements "x is a baby", "x is logical", "x is able to manage a crocodile" and "x is despised" respectively. Suppose that the domain consists of all people. Express the following using quantifiers and the above predicates: "Nobody is despised who can manage a crocodile".

   Solution from the text: $\forall x(R(x) \rightarrow \neg S(x))$.

4. Nested quantifiers

   Express the following using predicates, quantifiers, logical connectives and mathematical operators where the domain is all integers.

   (a) "The sum of squares of two integers is greater than or equal to the square of their sum."

   Solution:

   $$\forall x \forall y((x^2 + y^2) \geq (x + y)^2).$$

   (b) "The absolute value of the product of two integers equals the product of the abolute values of these integers."

   Solution:

   $$\forall x \forall y(|x \cdot y| = |x| \cdot |y|.$$

   (c) "The difference of two negative integers is not necessarily negative."

   Solution:

   $$\exists x \exists y((x < 0) \wedge (y < 0) \wedge (x - y \geq 0))$$

   Note:

   1. Some students translated this as $\forall x \exists y(x - y \geq 0)$, which is NOT the same statement but a stronger one. No marks were taken off as the statement was a stronger one that the question asks for but strictly speaking it is incorrect.

   2. Some students translated this as $\exists x \exists y((x < 0) \wedge (y < 0) \rightarrow (x - y \geq 0))$. The problem is again the distinction between $a \rightarrow b$ and $a \wedge b$. The former does not require that a be true, the latter does.

   (d) "The absolute value of the sum of two integers does not exceed the sum of the abolute values of these integers."

   Solution:

   $$\forall x \forall y(|x + y| \leq |x| + |y|)$$

   (e) Express the negative of the following statement so that all negation symbols immediately precede predicates.

   $\forall x \exists y(P(x, y) \rightarrow Q(x, y))$

Solution:

$$\begin{aligned}
\neg(\forall x \exists y(P(x,y) \rightarrow Q(x,y))) &\equiv \exists x \forall y \neg(P(x,y) \rightarrow Q(x,y)) && (0.3)\\
&\equiv \exists x \forall y \neg(\neg P(x,y) \vee Q(x,y)) && (0.4)\\
&\equiv \exists x \forall y P(x,y) \wedge \neg Q(x,y) && (0.5)
\end{aligned}$$

5. Express the negative of the following statement so that all negation symbols immediately precede predicates.

   $\forall x \exists y \exists z(T(x,y,z) \vee Q(x,y))$

   Solution:

   The negation is

$$\begin{aligned}
\neg(\forall x \exists y \exists z(T(x,y,z) \vee Q(x,y))) &\equiv \exists x \forall y \forall z \neg(T(x,y,z) \vee Q(x,y))\\
&\equiv \exists x \forall y \forall z \neg T(x,y,z) \wedge \neg Q(x,y))
\end{aligned}$$

6. Let $F(x,y)$ be the statement "x can fool y", where the domain consists of all people in the world. Use quantifiers to express the following statements

   (a) "Everyone can be fooled by somebody".
   Solution: $\forall x \exists F(y,x)$.

   (b) Let $F(x,y)$ be the statement "x can fool y", where the domain consists of all people in the world. Use quantifiers to express the following statement: "There is no one who can fool everybody".
   Solution: Translated literally the sentence is $\neg(\exists x \forall y \exists F(x,y))$. We can simplify this to $\forall x \exists y \neg F(x,y)$.

7. Express the following statement in predicate logic: "Every real number has exactly 2 square roots".

   Solution from the text: $\forall x \exists a \exists b(a \neq b \wedge \forall c(c^2 = x \leftrightarrow (c = a \vee c = b)))$

   Note: There are other possible solutions.

8. Express the negative of the following statement so that all negation symbols immediately precede predicates.

   $\forall x \exists y(P(x,y) \rightarrow Q(x,y))$

   Solution:

$$\begin{aligned}
\neg(\forall x \exists y(P(x,y) \rightarrow Q(x,y))) &\equiv \exists x \neg(\exists y(P(x,y) \rightarrow Q(x,y)))\\
&\equiv \exists x \forall y(\neg(P(x,y) \rightarrow Q(x,y)))\\
&\equiv \exists x \forall y(\neg(\neg P(x,y) \vee Q(x,y)))\\
&\equiv \exists x \forall y(P(x,y) \wedge \neg Q(x,y))
\end{aligned}$$

9. Inference.

Determine if each of these statements is correct or incorrect and explain why.

(a) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.

Solution: In questions like this one, you MUST define any propositions and/or predicates you use. So let us define $C(x)$ to be the predicate "x is a convertible", $F(x)$ be the predicate "x is fun to drive" and the domain be the set of all cars.

Then the information given is

$\forall x(C(x) \to F(x))$

$\neg C(IsaacsCar)$

So $\neg F(IsaacsCar)$.

First we infer that $C(IsaacsCar) \to F(IsaacsCar)$. Then we see that the conclusion is invalid.

(b) Quincy likes all action movies. Quincy likes the movie My Cousin Vinny. Therefore, My Cousin Vinny is an action movie (denying the hypothesis).

Solution:

Let us define $A(x)$ to be the predicate "x is an action movie", $L(x)$ be the predicate "Quincy likes x" and the domain be the set of all movies.

Then the information given is

$\forall x(A(x) \to L(x))$

$L(MyCousinVinny)$

So $A(MyCousinVinny)$.

First we infer that $A(MyCousinVinny) \to L(MyCousinVinny)$. Then we see that the conclusion is invalid (affirming the conclusion).

(c) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

Solution: Let us define $L(x)$ to be the predicate "x is a lobsterman", $S(x)$ be the predicate "x sets at leasy a dozen traps" and the domain be the set of all men.

Then the information given is

$\forall x(L(x) \to S(x))$

$L(Hamilton)$

So $S(Hamilton)$.

First we infer that $L(Hamilton) \to S(Hamilton)$. Then we see that the conclusion is valid (using modus ponens).

(d) Every CSE major takes discrete math. Natasha is taking discrete math. Therefore, Natasha is a CSE major.

Solution: In questions like this one, you MUST define any propositions and/or predicates you use. So let us define $C(x)$ to be the predicate "x is a CSE major", $D(x)$ be the predicate "x takes discrete math" and the domain be the set of York University students.

Then the information given is
$\forall x(C(x) \to D(x))$
$D(Natasha)$
So $C(Natasha)$.

First we infer that $C(Natasha) \to D(Natasha)$. Then we see that the conclusion is invalid (affirming the conclusion)

(e) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.

Solution: We define the predicate $P(x)$ as "x is a parrot", $F(x)$ as "x likes fruit" and the set of all birds to be the domain.

Then the information given is
$\forall x(P(x) \to F(x))$
$\neg P(MyPetBird)$
So $\neg F(MyPetBird)$.

First we infer that $P(MyPetBird) \to F(MyPetBird)$. Then we see that the conclusion is invalid (denying the hypothesis)

(f) Everyone who eats granola daily is healthy. Linda is not healthy. Therefore, Linda does not eat granola daily.

Solution:

We define the predicate $G(x)$ as "x eats granola daily", $H(x)$ as "x is healthy" and the set of all people to be the domain.

Then the information given is
$\forall x(G(x) \to H(x))$
$\neg H(Linda)$
So $\neg G(Linda)$.

First we infer that $G(Linda) \to H(Linda)$. Then we see that the conclusion is valid (Modus Tollens)

(g) Express using logical operators, quantifiers and predicates: "The conjunction of two tautologies is a tautology".

Solution from the text: Let $T(x)$ mean that x is a tautulogy. Then $\forall x \forall y((T(x) \wedge T(y)) \to T(x \wedge y)$.

Note: The solution should also mention that the domain is all propositions. Since x, y are propositions, $x \wedge y$ is well defined.

(h) Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \to \neg S(x))$ and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

Solution from the text:

1.	$\exists x \neg P(x)$	Premise
2.	$\neg P(c)$	Existential instantiation from (1)
3.	$\forall x(P(x) \vee Q(x))$	Premise
4.	$P(c) \vee Q(c)$	Existential instantiation from (3)
5.	$Q(c)$	Disjunctive Syllogism from (2) and (4)
6.	$\forall x(\neg Q(x) \vee S(x))$	Premise
7.	$\neg Q(c) \vee S(c)$	Universal instantiation from (6)
8.	$S(c)$	Disjunctive Syllogism from (5) and (7)
9.	$\forall x(R(x) \rightarrow \neg S(x))$	Premise
10.	$R(c) \rightarrow \neg S(c)$	Universal instantiation from (9)
11.	$\neg R(c)$	Modus Tollens from (8) and (10)
12.	$\exists x \neg R(x)$	Existential generalization from (11)

10. Proof by cases.

    Prove that $n, n+7$ cannot both be perfect cubes where n is an integer greater than 1.

    Solution: There are two cases. The first case – n is not a perfect cube - is trivial, because then we are done. The second case is $n > 1$ is a perfect cube, and we need to show $n + 7$ cannot be a perfect cube. Let $n = m^3$ for some positive integer m. Then we prove that $m^3 + 1$ cannot be a perfect cube.

    We prove this by contradiction. First we note that no two positive integers have the same cube, so we let $n + 1 = (m + k)^3$ where k is a non-zero positive integer.

$$\begin{aligned} n+7 &= m^3+7 \\ &= (m+k)^3, \text{ so} \\ m^3+7 &= (m+k)^3 \\ 7 &= 3m^2k+3mk^2+k^3 \end{aligned}$$

    Since $k \geq 1$ it follows that RHS $\geq 3m^2 + 3m + 1 = 7$ which is not possible since $m > 1$ implies $3m^2 + 3m + 1 \geq 7$.

11. Functions

    Define a function f from $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ as $f(m, n) = m^2 - 10$. Is this function onto? Prove your answer.

    Solution: First let us note that the function $f(m, n) = m^2$ is not onto, because many integers are not perfect squares, e.g. 5. Choose a prime number p and consider $p - 10$. No pair m, n can map to it because it it did, $m^2 = p$ which is a contradiction.

    Note: It is perfectly reasonable to define $f(m, n) = m^2$. Recall familiar functions like $f(m) = 1$ – it too ignores its input.

12. Prove the following statement: If a, b, c are odd integers, then $ax^2 + bx + c = 0$ does not have a rational number solution.

    Solution:

Suppose for the sake of contradiction that the statement is false. Then there is a rational number root p/q, i.e., $p, q \in \mathbb{N}, q \neq 0$. We can further assume that p, q have no common factors.

So $a(p/q)^2 + b(p/q) + c = 0$. Multiplying by q^2 throughout, we get

$$ap^2 + bpq + cq^2 = 0.$$

We need to consider four cases.

Case 1 (p, q both odd): In this case all three terms are odd and three odd numbers cannot sum to 0.

Case 2 (p odd, q even): In this case the first term is odd but the other two terms are even so that the sum must be odd and thus cannot be zero.

Case 3 (p even, q odd): In this case the first two terms are even but the third is odd, so that the sum must be odd and thus cannot be zero.

Case 4 (p, q both even): In this p, q must have a common factor of 2, which is not possible by our assumption.

Thus in all cases we get a contradiction. So it must be the case that the given equation has no rational roots.

13. Let $p < q$ be two consecutive odd primes. Prove that $p + q$ is a composite number, having at least three, not necessarily distinct, prime factors.

    Solution:

    This is another direct proof. Since p, q are *odd* primes, so $fracp + q2$ is an integer. Since p, q are *consecutive* primes and $p < \frac{p+q}{2} < q$, so $\frac{p+q}{2}$ is composite and must have at least two factors m, n. These facts imply that $2, m, n$ are factors of $p + q$.

14. A function $f(x)$ is said to be strictly increasing if $f(b) > f(a)$ for all $b > a$. Prove that a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.

    Solution: This is a very simple proof by contradiction. If f is not one-to-one, there exists x_1, x_2 such that $f(x_1) = f(x_2)$. Without loss of generality, assume $x_1 < x_2$. Then $f(x_1) < f(x_2)$, which contradicts the assumption $f(x_1) = f(x_2)$.