MATH/EECS 1028: Discrete Math for Engineers
Winter 2015
Tutorial 3 (Week of Jan 26, 2015)

Notes:

1. Assume $\mathbb{R}$ to denote the real numbers, $\mathbb{Z}$ to denote the set of integers $(\ldots, -2, -1, 0, 1, 2, \ldots)$ and $\mathbb{N}$ to denote the natural numbers $(1, 2, 3, \ldots)$.
2. Topics: Sequences, Logic.
3. Note to the TA: Attendance will be taken this week. No quiz this week.

Questions:

1. (a) Write down the truth table for the following proposition. Then indicate whether is a tautology, contradiction or neither.

$$(p \wedge q) \rightarrow (p \rightarrow q)$$

Solution: The truth table is given below. Since the last column has all T's, the given proposition is a tautology.

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

(b) Let p, q, r be the propositions:
p: You have the flu
q: You miss the final examination
r: You pass the course.
Write the following proposition as an English sentence.

$$(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$$

Solution: The English sentence is

If you have the flu, you will not pass the course OR if you miss the final examination, you will not pass the course.

NOTE: This is NOT logically equivalent to "If you have the flu OR if you miss the final examination, you will not pass the course"!
To see this let us observe that

$$(p \rightarrow \neg r) \vee (q \rightarrow \neg r) \equiv (p \wedge q) \rightarrow \neg r).$$

This fact (given on Table 7, page 25 in the text) can be proved using truth tables or by using rules for manipulation of logical expressions.

2. Form the contrapositive of these statements:

   (a) If you don't take the final examination, you will get an F for the course.

   (b) If a quadrilateral is a rectangle, it has 4 equal angles.

   (c) If a triangle has either two equal sides or two equal angles, then it is an isosceles triangle.

   Solution:

   (a) Let $p =$ you take the final examination, $q =$ you get an F for the course. Then the given assertion translates to

   $$\neg p \to q.$$

   The contrapositive of this implication is

   $$\neg q \to p.$$

   Translating back to English, we have the statement: *If you did not get an F for the course, you must have taken the final examination.*

   (b) Let $p =$ a quadrilateral is a rectangle, $q =$ it has 4 equal angles. Then the given assertion translates to

   $$p \to q.$$

   The contrapositive of this implication is

   $$\neg q \to \neg p.$$

   Translating back to English, we have the statement: *If it does not have four equal angles, then a quadrilateral is not a rectangle.*

   (c) Let $p =$ a triangle has either two equal sides or two equal angles $q =$ it is an isosceles triangle. Then the given assertion translates to

   $$p \to q.$$

   The contrapositive of this implication is

   $$\neg q \to \neg p.$$

   Translating back to English, we have the statement: *If it is not an isosceles triangle, then a triangle does not have two equal sides and it also does not have two equal angles.*

3. Decide whether the following statements are tautologies or contradictions or neither. Prove your answer in each case.

   (a) $(p \to q) \vee (q \to p)$.

   (b) $(p \wedge q) \vee (q \to \neg p)$.

(c) $(p \vee \neg q) \to (q \wedge \neg p)$.

Solution: The solution using truth tables is very easy. Here we provide a different solution.

(a)

$$\begin{aligned}(p \to q) \vee (q \to p) &\equiv (\neg p \vee q) \vee (\neg q \vee p)\\ &\equiv \neg p \vee q \vee \neg q \vee p\\ &\equiv \neg p \vee (q \vee \neg q) \vee p\\ &\equiv \neg p \vee TRUE \vee p\\ &\equiv TRUE\end{aligned}$$

Therefore, this is a tautology.

(b)

$$\begin{aligned}(p \wedge q) \vee (q \to \neg p) &\equiv (p \wedge q) \vee (\neg q \vee \neg p)\\ &\equiv (\neg q \vee \neg p \vee p) \wedge (\neg q \vee \neg p \vee q) \text{ Distributive law}\\ &\equiv (\neg q \vee TRUE) \wedge (TRUE \vee \neg p) \text{ Commutative and negation laws}\\ &\equiv TRUE\end{aligned}$$

Therefore, this is a tautology.

(c)

$$\begin{aligned}(p \vee \neg q) \to (q \wedge \neg p) &\equiv \neg(p \vee \neg q) \vee (q \wedge \neg p)\\ &\equiv (\neg p \wedge q) \vee (q \wedge \neg p)\\ &\equiv ((\neg p \wedge q) \vee q) \wedge ((\neg p \wedge q) \vee \neg p) \text{ Distributive law}\\ &\equiv q \wedge \neg p \text{ Absorption law}\end{aligned}$$

This is neither a tautology nor a contradiction - the last statement is true when$q$ is true and p is false and; the same statement is false when q is false.

4. Each argument below is either correct or it has a fallacy (but not both!). Write the argument in symbols and then determine whether the argument is valid. If it is valid, write whether it uses *modus ponens* or *modus tollens*.

   (a) If both numbers are even, then the sum is even. They are not both even. Therefore the sum is not even.

   (b) If this University is large, then it has large departments. This University has large departments. Therefore, it is large.

   Solution:

   (a) Let $p =$ Both numbers are even, $q =$ The sum is even. Then the given statements translate to

$$\begin{aligned}p &\to q\\ &\neg p\\ \therefore\ &\neg q\end{aligned}$$

This is invalid. The book calls this an inverse fallacy, also called “fallacy of denying the hypothesis”.

(b) Let $p =$ this University is large, $q =$ it has large departments.
Then the given statements translate to

$$\begin{array}{r} p \to q \\ q \\ \therefore p \end{array}$$

This is invalid. The book calls this a converse fallacy, also called “fallacy of affirming the conclusion”.

5. Find the sum of all integers between 1 and 100 that leave remainder 2 upon division by 6.

Solution: We want the sum of the integers of the form $6r + 2, r = 0, 1, \ldots, 16$. But this is $\sum_{r=0}^{16}(6r + 2) = 6\sum_{r=0}^{16} r + \sum_{r=0}^{16} 2 = 6 * 16 * 17/2 + *17 = 850$.

6. Find the sum

$$S = 5 + 55 + 555 + \ldots + 5 \ldots 5(\text{n 5's})$$

Solution: This one needs a clever trick. Note that if we multiply S by $9/5$ we get a series $9S/5 = 9 + 99 + 999 + \ldots + 9 \ldots 9(\text{n 9's})$. If we add 1 to each term we get a geometric series with $a = 10, r = 10$, so that the sum is $10(10^n - 1)/(10 - 1) = 10(10^n - 1)/9$. So we have $9S/5 + n = 10(10^n - 1)/9$.

7. The first four terms of an arithmetic sequence (in order) are $x + y, x - y, xy$ and x/y. What is the value of the fifth term?

Solution: Since this is an arithmetic sequence, every pair of adjacent terms must have the same common difference. The first pair has the common difference $-2y$. We also have

$$\begin{aligned} xy - (x - y) &= -2y \\ xy - x &= -3y \\ x &= \frac{-3y}{y - 1} \end{aligned}$$

Finally

$$\begin{aligned} x/y - xy &= -2y \\ x - xy^2 &= -2y^2 \\ x &= \frac{-2y^2}{1 - y^2} \end{aligned}$$

So we have

$$\frac{-3y}{y - 1} = \frac{-2y^2}{1 - y^2}.$$

Note that $y = 0$ is not possible (x/y must be defined) and $y = 1$ is not possible (the first two terms would be unequal but the last two terms would be equal). So we can cancel y from the numerators and $1 - y$ from the denominators.

So we have

$$\begin{aligned}
\frac{-3y}{y-1} &= \frac{-2y^2}{1-y^2} \\
\frac{-3}{y-1} &= \frac{-2y}{1-y^2} \\
\frac{-3}{-1} &= \frac{-2y}{1+y} \\
3+3y &= -2y \\
5y &= -3 \\
y &= -3/5 \\
x &= \frac{-3y}{y-1} \\
&= \frac{9/5}{-8/5} \\
&= -\frac{9}{8}
\end{aligned}$$

So the fifth term is $x/y - 2y = 15/8 + 6/5 = 3.075$.

8. Find a formula, in terms of n, for the sum of the first n terms of the sequence

$$1, 1+2, 1+2+2^2, 1+2+2^2+2^3, ...$$

Solution: Let us find a formula for the i^{th} term t_i, $i = 1, 2, ...$ From inspection,

$$\begin{aligned}
t_i &= \sum_{j=0}^{i-1} 2^j \\
&= \frac{2^i - 1}{2 - 1} \\
&= 2^i - 1
\end{aligned}$$

Therefore the sum we have to compute is

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n} t_i \\
&= \sum_{i=1}^{n} (2^i - 1) \\
&= \sum_{i=1}^{n} 2^i - \sum_{i=1}^{n} 1 \\
&= \frac{2(2^n - 1)}{2 - 1} - n \\
&= 2^{n+1} - n - 2
\end{aligned}
$$