Negation of Quantifiers

- "There is no student who can …"
- "Not all professors are bad…."
- "There is no Toronto Raptor that can dunk like Vince …"
- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$ why?
- $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$
- Careful: The negation of "Every Canadian loves Hockey" is NOT "No Canadian loves Hockey"! Many, many students make this mistake!

Nested Quantifiers

- Allows simultaneous quantification of many variables.
- E.g. – domain integers,
  - $\exists x \exists y \exists z \; x^2 + y^2 = z^2$ (Pythagorean triples)
  - $\forall n \exists x \exists y \exists z \; x^n + y^n = z^n$ (Fermat's Last Theorem implies this is false)
- Domain real numbers:
  - $\forall x \forall y \exists z \; (x < z < y) \lor (y < z < x)$ Is this true?
  - $\forall x \forall y \exists z \; (x=y) \lor (x < z < y) \lor (y < z < x)$
  - $\forall x \forall y \exists z \; (x \neq y) \rightarrow (x < z < y) \lor (y < z < x)$

Nested Quantifiers - 2

$\forall x \, \exists y \, (x + y = 0)$ is true over the integers

- Assume an arbitrary integer x.
- To show that there exists a y that satisfies the requirement of the predicate, choose y = -x. Clearly y is an integer, and thus is in the domain.
- So $x + y = x + (-x) = x - x = 0$.
- Since we assumed nothing about x (other than it is an integer), the argument holds for any integer x.
- Therefore, the predicate is TRUE.

Nested Quantifiers - 3

- **Analogy:** quantifiers are like loops:

  An inner quantified variable can depend on the outer quantified variable.

E.g. in $\forall x\ \exists y\ (x + y = 0)$ we chose y=-x, so for different x we need different y to satisfy the statement.

$\forall p\ \exists j$ Accept (p,j)   p,j have different domains

does NOT say that there is a j that will accept all p.

Nested Quantifiers - 4

- **Caution:** In general, order matters! Consider the following propositions over the integer domain:

  $\forall x\ \exists y\ (x < y)$ and $\exists y\ \forall x\ (x < y)$

- $\forall x\ \exists y\ (x < y)$: “there is no maximum integer”
- $\exists y\ \forall x\ (x < y)$: “there is a maximum integer”
- Not the same meaning at all!!!

Negation of Nested Quantifiers

- Use the same rule as before carefully.
- Ex 1: $\neg \exists x \forall y (x + y = 0)$
  - This is equivalent to $\forall x \neg \forall y (x + y = 0)$
  - This is equivalent to $\forall x \exists y \neg (x + y = 0)$
  - This is equivalent to $\forall x \exists y (x + y \neq 0)$
- Ex 2: $\neg \forall x \exists y (x < y)$
  - This is equivalent to $\exists x \neg \exists y (x < y)$
  - This is equivalent to $\exists x \forall y \neg (x < y)$
  - This is equivalent to $\exists x \forall y (x \geq y)$

Exercises

Check that:

- $\forall x \, \exists y \, (x + y = 0)$ is not true over the positive integers.
- $\exists x \, \forall y \, (x + y = 0)$ is not true over the integers.
- $\forall x <> 0 \, \exists y \, (y = 1/x)$ is true over the real numbers.

Readings and notes

- Read 1.4-1.5.
- Practice: Q2,8,16,30 (pg 65-67)

- Next: Rules of inference for quantified statements (1.6).

Inference rules for quantified statements

- Very intuitive, e.g. Universal instantiation – If $\forall x\, P(x)$ is true, we infer that $P(a)$ is true for any given a
- E.g.: Universal Modus Ponens:

  $\forall x\, P(x) \rightarrow Q(x)$ and $P(a)$ imply $Q(a)$

  If x is odd then x^2 is odd, a is odd. So a^2 is odd.
- Read rules on page 76
- Again, understanding is required, memorization is not.

Commonly used technique: Universal generalization

Prove: If x is even, x+2 is even

- Proof:

Prove: If x^2 is even, x is even

[Note that the problem is to prove an implication.]

- Proof: if x is not even, x is odd. Therefore x^2 is odd. This is the contrapositive of the original assertion.

Aside: Inference and Planning

- The steps in an inference are useful for planning an action.
- Example: your professor has assigned reading from an out-of-print book. How do you do it?
- Example 2: you are participating in the television show “Amazing race”. How do you play?

Aside 2: Inference and Automatic Theorem-Proving

- The steps in an inference are useful for proving assertions from axioms and facts.
- Why is it important for computers to prove theorems?
  - Proving program-correctness
  - Hardware design
  - Data mining
  - …..

Aside 3: Inference and Automatic Theorem-Proving

- Sometimes the steps of an inference (proof) are useful. E.g. on Amazon book recommendations are made.
- You can ask why they recommended a certain book to you (reasoning).

Next

- Introduction to Proofs (Sec 1.7)
- What is a (valid) proof?
- Why are proofs necessary?

Introduction to Proof techniques

Why are proofs necessary?

What is a (valid) proof?

What details do you include/skip?

"Obviously", "clearly"…