Tools for reasoning: Logic

Ch. 1: Introduction to Propositional Logic

- Truth values, truth tables
- Boolean logic: $\vee \quad \wedge \quad \neg$
- Implications: $\rightarrow \quad \leftrightarrow$

Why study propositional logic?

- A formal mathematical “language” for precise reasoning.
- Start with propositions.
- Add other constructs like negation, conjunction, disjunction, implication etc.
- All of these are based on ideas we use daily to reason about things.

Propositions

- Declarative sentence
- **Must** be either True or False.

Propositions:

- York University is in Toronto
- York University is in downtown Toronto
- All students at York are Computer Sci. majors

Not propositions:

- Do you like this class?
- There are x students in this class.

Propositions - 2

- Truth value: True or False
- Variables: p,q,r,s,…
- Negation:
- $\neg p$ (“not p”)
- Truth tables

p	$\neg p$
T	F
F	T

Caveat: negating propositions

$\neg$p: “it is not the case that p is true”

p: “it rained more than 20 inches in TO”

p: “John has many iPads”

Practice: Questions 1-7 page 12.

Q10 (a) p: “the election is decided”

Conjunction, Disjunction

- Conjunction: $p \wedge q$ [“and”]
- Disjunction: $p \vee q$ [“or”]

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Examples

Q11, page 13

p: It is below freezing

q: It is snowing

(a) It is below freezing and snowing

(b) It is below freezing but not snowing

(d) It is either snowing or below freezing (or both)

Exclusive OR (XOR)

- $p \oplus q$ – T if p and q have different truth values, F otherwise
- Colloquially, we often use OR ambiguously – “an entrée comes with soup or salad” implies XOR, but “students can take MATH XXXX if they have taken MATH 2320 or MATH 1019” usually means the normal OR (so a student who has taken both is still eligible for MATH XXXX).

Conditional

- $p \rightarrow q$ ["if p then q"]
- p: *hypothesis*, q: *conclusion*
- E.g.: "If you turn in a homework late, it will not be graded"; "If you get 100% in this course, you will get an A+".
- TRICKY: Is $p \rightarrow q$ TRUE if p is FALSE?

  YES!!
- Think of "If you get 100% in this course, you will get an A+" as a promise – is the promise violated if someone gets 50% and does not receive an A+?

Conditional - 2

- $p \rightarrow q$ ["if p then q"]
- Truth table:

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Note the truth table of $\neg p \vee q$

Logical Equivalence

- $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent
- Truth tables are the simplest way to prove such facts.
- We will learn other ways later.

Contrapositive

- Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- Any conditional and its contrapositive are logically equivalent (have the same truth table) – Check by writing down the truth table.
- E.g. The contrapositive of “If you get 100% in this course, you will get an A+” is “If you do not get an A+ in this course, you did not get 100%”.

E.g.: Proof using contrapositive

Prove: If x^2 is even, x is even

- Proof 1: $x^2 = 2a$ for some integer a. Since 2 is prime, 2 must divide x.
- Proof 2: if x is not even, x is odd. Therefore x^2 is odd. This is the contrapositive of the original assertion.

Converse

- Converse of $p \rightarrow q$ is $q \rightarrow p$
- Not logically equivalent to conditional
- Ex 1: “If you get 100% in this course, you will get an A+” and “If you get an A+ in this course, you scored 100%” are not equivalent.
- Ex 2: If you won the lottery, you are rich.

Other conditionals

Inverse:

- inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$
- How is this related to the converse?

Biconditional:

- "If and only if"
- True if p,q have same truth values, false otherwise. Q: How is this related to XOR?
- Can also be defined as $(p \rightarrow q) \wedge (q \rightarrow p)$

Example

- Q16(c) 1+1=3 if and only if monkeys can fly.

Readings and notes

- Read pages 1-12.
- Think about the notion of truth tables.
- Master the rationale behind the definition of conditionals.
- Practice translating English sentences to propositional logic statements.

Next

Ch. 1.2, 1.3: Propositional Logic - contd

- Compound propositions, precedence rules
- Tautologies and logical equivalences
- Read only the first section called "Translating English Sentences" in 1.2.

Compound Propositions

- Example: $p \wedge q \vee r$: Could be interpreted as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$
- precedence order: $\neg \ \wedge \ \vee \ \rightarrow \ \leftrightarrow$ (IMP!) (Overruled by brackets)
- We use this order to compute truth values of compound propositions.

Tautology

- A compound proposition that is always TRUE, e.g. $q \vee \neg q$
- Logical equivalence redefined: p,q are logical equivalences if $p \leftrightarrow q$ is a tautology. Symbolically $p \equiv q$.
- Intuition: $p \leftrightarrow q$ is true precisely when p,q have the same truth values.

Manipulating Propositions

- Compound propositions can be simplified by using simple rules.
- Read page 25 - 28.
- Some are obvious, e.g. Identity, Domination, Idempotence, double negation, commutativity, associativity
- Less obvious: Distributive, De Morgan’s laws, Absorption

Distributive Laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Intuition (not a proof!) – For the LHS to be true: p must be true and q or r must be true. This is the same as saying p and q must be true or p and r must be true.

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Intuition (less obvious) – For the LHS to be true: p must be true or both q and r must be true. This is the same as saying p or q must be true and p or r must be true.

Proof: use truth tables.

De Morgan's Laws

$\neg(q \vee r) \equiv \neg q \wedge \neg r$

Intuition – For the LHS to be true: neither q nor r can be true. This is the same as saying q and r must be false.

$\neg(q \wedge r) \equiv \neg q \vee \neg r$

Intuition – For the LHS to be true: $q \wedge r$ must be false. This is the same as saying q or r must be false.

Proof: use truth tables.

Using the laws

- Q: Is $p \rightarrow (p \rightarrow q)$ a tautology?
- Can use truth tables
- Can write a compound proposition and simplify

Inference in Propositional Logic

- in Section 1.6 pages 71-75
- Recall: the reason for studying logic was to formalize derivations and proofs.
- How can we infer facts using logic?
- Simple inference rule (Modus Ponens) :
  From (a) $p \rightarrow q$ and (b) p is TRUE,
  we can infer that q is TRUE.

Modus Ponens continued

Example:

(a) if these lecture slides (ppt) are online then you can print them out

(b) these lecture slides are online

Can you print out the slides?

- Similarly, From $p \rightarrow q$, $q \rightarrow r$ and p is TRUE, we can infer that r is TRUE.

Inference rules - continued

- $((p \rightarrow q) \wedge p) \rightarrow q$ is a TAUTOLOGY.
- Modus Tollens, Hypothetical syllogism and disjunctive syllogism can be seen as alternative forms of Modus Ponens
- Other rules like

"From p is true we can infer $p \vee q$" are very intuitive

Inference rules - continued

Resolution: From

(a) $p \vee q$ and

(b) $\neg p \vee r$, we can infer that

$q \vee r$

<u>Exercise:</u> check that

$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

is a TAUTOLOGY.

Very useful in computer generated proofs.

Inference rules - continued

- Read rules on page 72.
- Understanding the rules is crucial, memorizing is not.
- You should be able to see that the rules make sense and correspond to our intuition about formal reasoning.

Limitations of Propositional Logic

- What can we NOT express using predicates?

  Ex: How do you make a statement about all even integers?

  If $x > 2$ then $x^2 > 4$

- A more general language: Predicate logic (Sec 1.4)