Homework Assignment #7
Due: July 15, 2015 at 7:00 p.m.

1. In this question, we consider a variant of a Turing machine called a Card Turing Machine (CTM). Instead of a tape, this machine has two piles of paper cards, which we call the left pile and the right pile. Each card has one character (from a finite card alphabet) written on it.

In each step, the CTM decides what to do next based on the current state of the machine and the characters written on the top card on each of the two piles. The step can change the state of the machine and either remove the top card from the left or right pile, or add a new card (with any desired character) to the top of the left or right pile. We assume that the bottom card of each pile has a special character $\perp$ written on it, and the machine is never allowed to remove a $\perp$ card. (This ensures that there is always at least one card on each pile.) There is no limit on how many cards the machine can use.

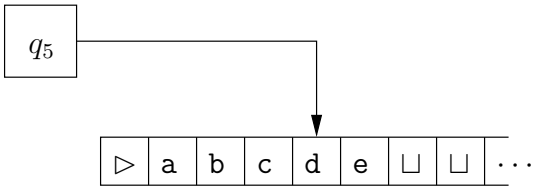
For example, if the machine is in state q_7 and the top two cards on the left and right pile are `b` and `c`, respectively, then the transition function of the Turing machine might tell it to change to state q_8 and remove the top card from the left pile.

To run the CTM on an input string $x = x_1x_2 \dots x_n$, the CTM begins in its initial state with one card on the left pile (containing the character $\perp$) and $n+1$ cards on the right pile, containing the characters $\perp, x_n, x_{n-1}, \dots, x_2, x_1$ (from bottom to top). The machine then runs according to its transition function until it enters its accepting or rejecting state.

(a) An ordinary Turing machine is defined formally in Definition 3.3 of the textbook. Give a similar definition for a CTM.

(b) Describe, in English, how you could simulate an ordinary Turing machine using a CTM. Your answer should include the following components.

- An explanation of how the CTM represents the information on an ordinary Turing machine's tape using the piles of cards. Try to make it as simple as possible.
- A picture of the configuration of your CTM that would correspond to the following configuration of the ordinary TM that it is simulating.



- A description, in English, of how the CTM would simulate each step of a Turing machine. (Your description should be general enough to cover any possible step of the simulated Turing machine.)

2. Suppose that L_1 and L_2 are decidable languages. Then there are algorithms DECIDEL$_1$ and DECIDEL$_2$ that decide whether a given string is in L_1 or L_2, respectively.

(a) Write an algorithm (in pseudocode) that decides $L_1 \cdot L_2$. Your algorithm should use DECIDEL$_1$ and DECIDEL$_2$ as subroutines.

(b) If DECIDEL$_1$ takes $O(n)$ steps to decide whether a given string of length n is in L_1 and DECIDEL$_2$ takes $O(n^2)$ steps to decide whether a given string of length n is in L_2, give a good upper bound on the number of steps your algorithm requires to test whether a string of length n is in $L_1 \cdot L_2$. Explain why your answer is correct.