Homework Assignment #7
Due: Thursday, November 6, 2014 at 4:00 p.m.

1. If M is a Turing machine with input alphabet Σ and $x \in \Sigma^*$ is an input string, let $time(M, x)$ be the number of steps that M takes on input string x before halting. (If M never halts on input x, then we define $time(M, x) = \infty$.) The *worst-case running time* of M on inputs of length n is the maximum number of steps M takes on any input of length n. More formally, $worst_M(n) = \max\{time(M, x) : x \in \Sigma^* \text{ and } |x| = n\}$.

Recall that the textbook provides a high-level description of a Turing machine M_3 that decides the language $C = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k : i \cdot j = k \text{ and } i, j, k \geq 1\}$ on page 174 (or page 146 of the second edition of the textbook). It is easy to see that the worst-case running time of that machine M_3 on inputs of length n is at least $\frac{1}{8} \cdot n^2$. (Just think about how the machine behaves on the input string $\mathtt{ab}^{n/2}\mathtt{c}^{n/2-1}$.)

Your task for this problem: Give a high-level description of a multitape Turing machine M' that decides the language C more efficiently. Your description should be at the level of detail given for M_3 on page 174 of the textbook. There should be a constant k such that the worst-case running time of M' on inputs of length n is at most $k \cdot n$. (Note that for large n, $k \cdot n$ is much smaller than $\frac{1}{8} \cdot n^2$.)

2. If L is a language over the alphabet Σ,

$$PREFIX(L) = \{x : \exists y \in \Sigma^* \text{ such that } xy \in L\}.$$

Prove that if L is recognizable, then $PREFIX(L)$ is also recognizable.
Note: for this question, you may use the Church-Turing thesis freely.