Homework Assignment #10
Due: Thursday, November 27, 2014 at 4:00 p.m.

1. Recall the definition of big-O notation. If f and g are functions from natural numbers to natural numbers, then we say f is $O(g)$ if there exist constants c, n_0 such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$.

If M is a Turing machine and w is an input string for it, let $T(M, w)$ be the number of steps that Turing machine M takes on input string w before halting. If M does not halt on input w, then define $T(M, w)$ to be ∞. The worst-case running time of M on inputs of size n is defined to be

$$Worst_M(n) = \max\{T(M, z) : z \text{ is an input string with } |z| = n\}.$$

Define a Turing machine M to be *fast* if its worst-case running time on inputs of size n is $O(n^3)$. In other words, M is fast if the function $Worst_M(n)$ is $O(n^3)$. Let $FAST_{TM} = \{\langle M\rangle : M \text{ is a fast Turing machine}\}$.

Given a Turing machine M and an input string w, define $M'_{M,w}$ be the Turing machine that does the following three things when it is run on input string z.
1. write blanks over the entire input string z and move back to the square after the $\triangleright$
2. write w on the tape and move back to the square after the $\triangleright$
3. run M

(Note that M' is constructed so that, on any input string z, M' simply runs M on w.)

(a) Give a formula for $T(M'_{M,w}, z)$. You may use any of the functions defined above in your formula.

(b) If M halts on input w, show that there is a constant c such that for any input string z with $|z| \geq \sqrt[3]{T(M, w)}$, $M'_{M,w}(z)$ halts within $c \cdot |z|^3$ steps.

(c) **Use the definition** of big-O notation to prove that if M halts on input w, then $M'_{M,w}$ is fast.

(d) Suppose we had an algorithm $DecideFAST_{TM}(\langle M\rangle)$ that decides $FAST_{TM}$. Use it to build the following algorithm.

$MysteryAlgorithm(\langle M, w\rangle)$
    construct a machine $M'_{M,w}$ as described above
    output $DecideFAST(\langle M'\rangle)$
end $MysteryAlgorithm$

What language does $MysteryAlgorithm$ decide? Prove your answer is correct.

(e) Prove that $FAST_{TM}$ is undecidable.