SC/MATH 1090

11- First Order Semantics

Ref: G. Tourlakis, *Mathematical Logic*, John Wiley & Sons, 2008.

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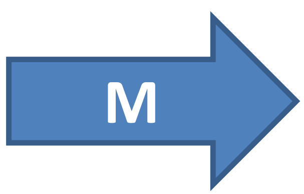
Overview

- Interpretations and domains
- Logically valid formulae
  - How does that relate to tautologies?
- Soundness and completeness in first order logic

Interpretations

- An interpretation, $\mathfrak{D}$= (D,M), translates a formula A to $A^{\mathfrak{D}}$.
- The two components of a first-order language interpretation are:
  - Domain D (non empty set, e.g. Natural numbers)
  - Translator M (a mapping)



$\top$,$\bot$	$\top^{\mathfrak{D}}$ is t and $\bot^{\mathfrak{D}}$ is f
p,q,…	$p^{\mathfrak{D}}$ is t or f
x,y,…	$x^{\mathfrak{D}}$ is a member of D
c ,…	$c^{\mathfrak{D}}$ is a member of D
f, …	$f^{\mathfrak{D}}$ specific function applicable to objects in D
φ,…	$\varphi^{\mathfrak{D}}$ specific predicate applicable to objects in D
everything else	unchanged

Interpretations (2)

- Given an interpretation $\mathfrak{D}$= (D,M), the translation of A, i.e. $A^{\mathfrak{D}}$ is obtained by:
  - Replacing $(\forall x)$ with $(\forall x \in D)$
  - Keep any bound object variables unchanged
  - Applying M to every other substrings of A

- Examples:

  If D={1,2,3}, and A is x=y, then $A^{\mathfrak{D}}$ is $x^{\mathfrak{D}}=y^{\mathfrak{D}}$

  Therefore, if $x^{\mathfrak{D}}$ is 2 and $y^{\mathfrak{D}}$ is 3, then $A^{\mathfrak{D}}$ is **f**

  If A is $(\forall x)\ x=y$ then $A^{\mathfrak{D}}$ is $(\forall x \in D)\ x=3$, which is **f** again

  If A is $(\forall x)\ \varphi(x,y)$, and $\varphi^{\mathfrak{D}}$ is $\leq$, then $A^{\mathfrak{D}}$ is $(\forall x \in D) x \leq 3$, which is **t**

Partial Translation of Formula

- A partial translation of formula A by $\mathfrak{D}$ with respect to the variables $x_1,\ldots,x_n$, is denoted by $A^{\mathfrak{D}}_{x_1,\ldots,x_n}$ and refers to translating A while leaving $x_1,\ldots,x_n$ untranslated.

- By above definition, $((\forall x)A)^{\mathfrak{D}}$ is $(\forall x \in D)A^{\mathfrak{D}}_{x}$

Logically Valid Formulae

- **Definition. (Model)** If $A^{\mathfrak{D}}$ is **t** for some *A* and $\mathfrak{D}$, in other words *A is true in the interpretation* $\mathfrak{D}$, then $\mathfrak{D}$ is a *model* of A, and is denoted by

$$\models_{\mathfrak{D}} A$$

- **Definition. (Universally, Logically, or Absolutely Valid formulae)** A formula *A* in first order logic is **valid** iff **every** interpretation $\mathfrak{D}$ is a model of A. This is denoted by

$$\models A$$

Tautology vs. Valid

- If $\models_{\mathrm{taut}} A$, then $\models A$.

- If $\models A$, it does NOT imply $\models_{\mathrm{taut}} A$.
  - Example: if A is x=x.

Soundness and Completeness

- **Metatheorem. (Soundness in 1st order logic)**

  If $\vdash A$ then $\models A$

- **Metatheorem. (Gödel's Completeness Theorem)**

  If $\models A$ then $\vdash A$