SC/MATH 1090

9- Generalization and Leibniz Rules

Ref: G. Tourlakis, *Mathematical Logic*, John Wiley & Sons, 2008.

York University

Department of Computer Science and Engineering

Overview

- Insert/Remove the universal quantifier
  - Weak generalization Metatheorem
  - Specialization rule
- The Leibniz Rules
  - Boolean Leib (BL)
  - Weak Leib (unconditional)
  - Strong Leib (conditional)

Weak Generalization

- **Metatheorem. (Weak Generalization)** If $\Gamma \vdash A$ and moreover x does not occur free in any formula in the set Γ, then $\Gamma \vdash (\forall x)$ A.
  - Note x can be free in A.
  - Proof by induction on length of a Γ-proof of A in Logic(2).

- **Corollary**. If $\Gamma \vdash A$ and moreover x does not occur free in *any formula used in the proof*, then $\Gamma \vdash (\forall x)$ A.

- **Corollary**. If $\vdash A$, then $\vdash (\forall x) A$.

NEVER Strong Generalization!

- NEVER Correct:

  Strong generalization: $A \vdash (\forall x)A$

- To be able to generalize A, we must have a proof for A.
- If A is an assumption, and not proven, we are not allowed to generalize.
- Strong generalization is SO WRONG that to show a formula X is not provable, it is sufficient to show that X can be used to prove strong generalization (Chapter 8).

Specialization Rule

- ***Metatheorem*. (Specialization Rule)** $(\forall x)A \vdash A[x:=t]$
  - Note: The above rule is applicable ONLY if the substitution is defined

- **Corollary**. $(\forall x)A \vdash A$

Remove/ Insert ($\forall x$)

- A simple template for some Hilbert proofs in 1st order Logic:

  1. Use **spec** to remove the universal quantifiers in the assumptions.

  2. Use simple Boolean Logic techniques we learned in previous chapters.

  3. Use (weak) **gen** to insert back the universal quantifiers provided gen **CONDITION**.

Examples

- $(\forall x)(\forall y)A \equiv (\forall y)(\forall x)A$
- $(\forall x)(A \wedge B) \equiv (\forall x)A \wedge (\forall x)B$ Distributivity of $\forall$ over $\wedge$
- **Metatheorem. ($\forall$-monotonicity)** *Provided x dnof in any formula in Γ,* if $\Gamma \vdash A \to B$ then $\Gamma \vdash (\forall x)A \to (\forall x)B$.
  - **Corollary**. If $\vdash A \to B$ then $\vdash (\forall x)A \to (\forall x)B$.
  - **Corollary**. If $\Gamma \vdash A \equiv B$ then $\Gamma \vdash (\forall x)A \equiv (\forall x)B$, *provided x dnof in any formula in Γ.*
  - **Corollary**. If $\vdash A \equiv B$ then $\vdash (\forall x)A \equiv (\forall x)B$.

Weak Leibniz (WL)

- **Metatheorem. (Weak Leibniz)**
  If $\vdash A \equiv B$, then $\vdash C[p\backslash A] \equiv C[p\backslash B]$.
  – Proof by induction on the complexity of C.
  – Also called "Weak Leib with unconditional substitution“
  – Note we DONOT have Strong Leib with unconditional substitution:
    $A \equiv B \vdash C[p\backslash A] \equiv C[p\backslash B]$ (WRONG!)
- **Corollary. (A more generous WL)**
  If $\Gamma \vdash A \equiv B$ and *none of the bound variables of C occur free in formulae of* Γ, then $\Gamma \vdash C[p\backslash A] \equiv C[p\backslash B]$.

Strong Leibniz (SL)

- **Metatheorem. (Strong Leibniz)** $A \equiv B \vdash C[p:=A] \equiv C[p:=B]$
  - Proof by induction on the complexity of C.
  - Also called "Strong Leib with conditional substitution"
- Example: $D \rightarrow (A \equiv B) \vdash D \rightarrow (C[p:=A] \equiv C[p:=B])$
- Exercise: $D \circ (A \equiv B) \vdash D \circ (C[p:=A] \equiv C[p:=B])$

Important Tools

- $\vdash (\forall x)(A \rightarrow B) \equiv A \rightarrow (\forall x)B$, provided x dnof in A.
- $\vdash (\forall x)(A \lor B) \equiv A \lor (\forall x)B$, provided x dnof in A.
- $\vdash (\exists x)(A \land B) \equiv A \land (\exists x)B$, provided x dnof in A.

Examples

1) $(\forall x)(\bot \rightarrow A) \equiv \top$ — Empty range

2) $\vdash (\forall x)\ (x{=}t \rightarrow A) \equiv A[x:=t]$ — provided x not free in t
One point rule

3) $\vdash (\exists x)\ (x{=}t \wedge A) \equiv A[x:=t]$ — provided x not free in t
One point rule- $\exists$ -version

Examples (2)

4) $\vdash (\forall x)(A \to B) \wedge (\forall x)(A \to C) \equiv (\forall x)(A \to B \wedge C)$

- Or $\vdash (\forall x)_A B \wedge (\forall x)_A C \equiv (\forall x)_A (B \wedge C)$

- $\exists$- version (dual of above):

5) $\vdash (\exists x)(A \wedge B) \vee (\exists x)(A \wedge C) \equiv (\exists x)(A \wedge (B \vee C))$

- Or $\vdash (\exists x)_A B \vee (\exists x)_A C \equiv (\exists x)_A (B \vee C)$

Examples (3)

6)	$\vdash (\forall x)_{A \vee B} C \equiv (\forall x)_A C \wedge (\forall x)_B C$	Range Split
7)	$\vdash (\forall x)_A (\forall y)_B C \equiv (\forall y)_B (\forall x)_A C$	provided y not free in A and x not free in B Interchange of dummies
8)	$\vdash (\exists x)_A (\exists y)_B C \equiv (\exists y)_B (\exists x)_A C$	provided y not free in A and x not free in B Dual of above
9)	$(\forall x)\ (\forall y)_{A \wedge B} C \equiv (\forall x)_A\ (\forall y)_B C$	provided y not free in A Nesting
10)	$(\exists x)\ (\exists y)_{A \wedge B} C \equiv (\exists x)_A\ (\exists y)_B C$	provided y not free in A Dual of above

Dummy Renaming

- **Theorem. (Dummy Renaming for $\forall$)** If z does not occur in A (free nor bound), then $\vdash (\forall x)A \equiv (\forall z)A[x:=z]$.

- **Theorem. (Dummy Renaming for $\exists$)** If z does not occur in A (free nor bound), then $\vdash (\exists x)A \equiv (\exists z)A[x:=z]$.