SC/MATH 1090

7- Boolean Semantics

Ref: G. Tourlakis, *Mathematical Logic*, John Wiley & Sons, 2008.

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Overview

- Two main theorems:
  - Soundness: Our Boolean Logic is sound and truthful. Everything we can prove using the Boolean Logic is actually true.
  - Completeness: Our Boolean Logic is complete. Everything that is true (and can be represented in Boolean logic), the Boolean Logic can prove.

Soundness

- The primary rules of inference are truthful, i.e.

$$A, A \equiv B \models_{taut} B$$

$$A \equiv B \models_{taut} C[\mathrm{p} := A] \equiv C[\mathrm{p} := B]$$

- All logical axioms are tautologies.

- **Metatheorem. (Soundness of Propositional Calculus)** If $\Gamma \vdash A$ then $\Gamma \models_{taut} A.$
  - Proof by induction on length of –proofs where A occurs.

- **Corollary**. If $\vdash A$, then $\models_{taut} A.$

Counter-example construction

- Soundness Theorem:
  - If $\Gamma \vdash A$, then $\Gamma \models_{taut} A.$
- *Contrapositive* of Soundness theorem:
  - If $\Gamma \not\models_{\mathrm{taut}} A$, then $\Gamma \not\vdash A$
- Reminder: $\Gamma \vdash A$ is a theorem schema.
- In order to show that A is not provable, we can find a specific formula, and some state v for which v(A)=**f**.

Completeness

- **Metatheorem**. **(Post's Tautology Theorem)**

  If $\Gamma \models_{taut} A$, then $\Gamma \vdash A$.

- *Contrapositive* of Post theorem:

  if $\Gamma \nvdash A$, then $\Gamma \not\models_{\mathrm{taut}} A$.