SC/MATH 1090

6- Deduction theorem & Resolution

Ref: G. Tourlakis, *Mathematical Logic*, John Wiley & Sons, 2008.

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Overview

- Deduction Theorem
  - Proving a Lemma (by induction) to prove deduction theorem (by induction on the length of proof!)
  - Why is deduction theorem important?
- Resolution- Easy life!!!
  - Definitions
  - Procedure for resolution
  - Examples

The Deduction Theorem

- **Lemma**. $A \rightarrow (B \equiv C) \vdash A \rightarrow (D[\mathbf{p}:=B] \equiv D[\mathbf{p}:=C])$
- **Metatheorem. (Deduction Theorem)** If $\Gamma \cup \{A\} \vdash B$, then also $\Gamma \vdash A \rightarrow B$.
- **Corollary**. $\Gamma + A \vdash B$ **iff** $\Gamma \vdash A \rightarrow B$.
- Why is it so important?
  - Theory: All theorems are equivalent to absolute theorems.
  - *Logic users: when proving $\Gamma \vdash A \rightarrow B$, instead prove $\Gamma + A \vdash B$ (which is usually easier)*

Equivalent statements

- **Metatheorem**. The following are equivalent:
  1) $\Gamma \vdash \bot$
  2) For all A, $\Gamma \vdash A$
  3) For some B, we have $\Gamma \vdash B \wedge \neg B$

  Γ is called **inconsistent** if any of above is proven for Γ.

- **Corollary. (Proof by Contradiction)** $\Gamma \vdash A$ **iff** $\Gamma + \neg A \vdash \bot$.
  – Logic users: when proving $\Gamma \vdash A$, instead prove $\Gamma + \neg A \vdash \bot$ (prove inconsistency if $\neg A$ is assumed)

Definitions

- **Definition. (Literal)** Atomic formulae and their negation, also formula variables, and their negation are **literals**.
- Examples of literals:

  $p, \neg p, \neg \bot, A, \neg B$

- **Definition. (Clause)** A **clause** is a *disjunction* of literals.
- Examples of clauses:

  $p \vee \neg p \quad , \quad \neg \bot \vee A \vee \neg B$

Resolution (1)

1) Using ***Deduction*** and ***proof by contradiction*** to move everything to the assumption side, and have $\perp$ on the conclusion side

- If Γ+A $\vdash$B then $\Gamma \vdash$A $\rightarrow$ B. **Deduction Theorem**
- If $\Gamma + \neg$ A $\vdash \perp$ then $\Gamma \vdash$ A. **Proof by contradiction**

In other words, instead of proving $\Gamma \vdash$A $\rightarrow$ B, it is sufficient to prove Γ + {A , $\neg$ B} $\vdash \perp$.

2) Remove any $\equiv$ using ***Ping-pong***

- A $\equiv$ B $\equiv$ (A $\rightarrow$ B) $\wedge$ (B $\rightarrow$ A) **Ping-pong theorem**

Resolution (2)

3) Remove any $\rightarrow$ using ***implication theorem***
   - $A \rightarrow B \equiv \neg A \vee B$ — Implication theorem

4) Move negation inwards using ***de Morgan***
   - $\neg(A \wedge B) \equiv \neg A \vee \neg B$ — de Morgan 1
   - $\neg(A \vee B) \equiv \neg A \wedge \neg B$ — de Morgan 2

5) ***Distribute*** $\vee$ over $\wedge$
   - $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ — Distributivity of $\vee$ over $\wedge$

Resolution (3)

6) ***Split*** $\wedge$

- $A \wedge B \vdash A$ and $A \wedge B \vdash B$ **Split**

7) Use ***Cut Rule*** to prove $\perp$

- $A \vee B,\ \neg A \vee C\ \vdash B \vee C$ **Cut Rule**
- $A, \neg A\ \vdash \perp$