SC/MATH 1090

2- Induction and properties of WFF

Ref: G. Tourlakis, *Mathematical Logic*, John Wiley & Sons, 2008.

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Overview

- **Simple** induction on natural numbers
- **Complete or Strong** induction
- Induction on **complexity** of WFF
- A few theorems about formulae

Simple Induction
on Natural Numbers

- P(n): Some property of the natural number n
- Goal: Prove that P(n) holds for all $n \in N$ (or prove P(n) is true for arbitrary *n*)
- Induction:
  - **Basis**: Prove that P(0) holds
  - **Induction Step**:

    Assume **Induction Hypothesis** (I.H.)

    P(k) holds for k=n-1

    then prove P(n) holds

Example
Simple induction on natural numbers

- Prove $$0+1+\dots+(n-1)+n=\frac{n(n+1)}{2}$$
$$P(n):\sum_{i=0}^{n} i=\frac{n(n+1)}{2}$$

- **Basis**: Prove P(0) holds $$P(0):\sum_{i=0}^{0} i=0=\frac{0.(0+1)}{2}$$

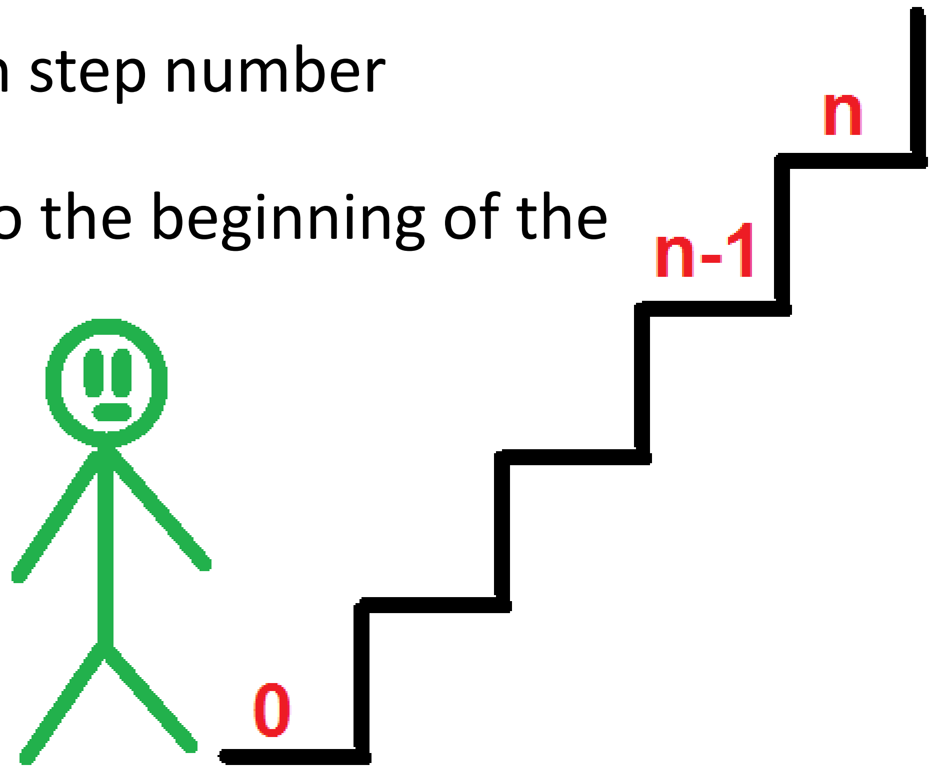
- **Induction Step:**
  - We assume P(k) holds for k=n-1:
  - Now we prove P(n) holds $$P(k=n-1):\sum_{i=0}^{k=n-1} i=\frac{k(k+1)}{2}=\frac{(n-1)n}{2}$$

$$P(n):\sum_{i=0}^{n} i=n+\sum_{i=0}^{n-1} i=n+\frac{(n-1)(n)}{2}=\frac{2n+n^2-n}{2}=\frac{n(n+1)}{2}$$

Example
Simple Induction on step number!

- Prove that robot R can go up the staircase to any arbitrary step
- Proof by simple induction on step number
- **Basis**: prove that R can get to the beginning of the staircase (step 0)
- **Induction step:**

Prove that R can take a step up
(If R can get to step (n-1),
it can go to step n)



Complete (strong) Induction

on Natural Numbers

- P(n): Some property of the natural number n
- Goal: Prove that P(n) holds for all $n \in N$ (or prove P(n) is true for arbitrary *n*)
- Induction:
  - **<u>Basis</u>**: Prove that P(0) holds
  - **<u>Induction Step</u>**:
    Assume **Induction Hypothesis** (I.H.)
    P(k) holds **for all k<n**
    then prove P(n) holds

Example
Strong induction on step number!

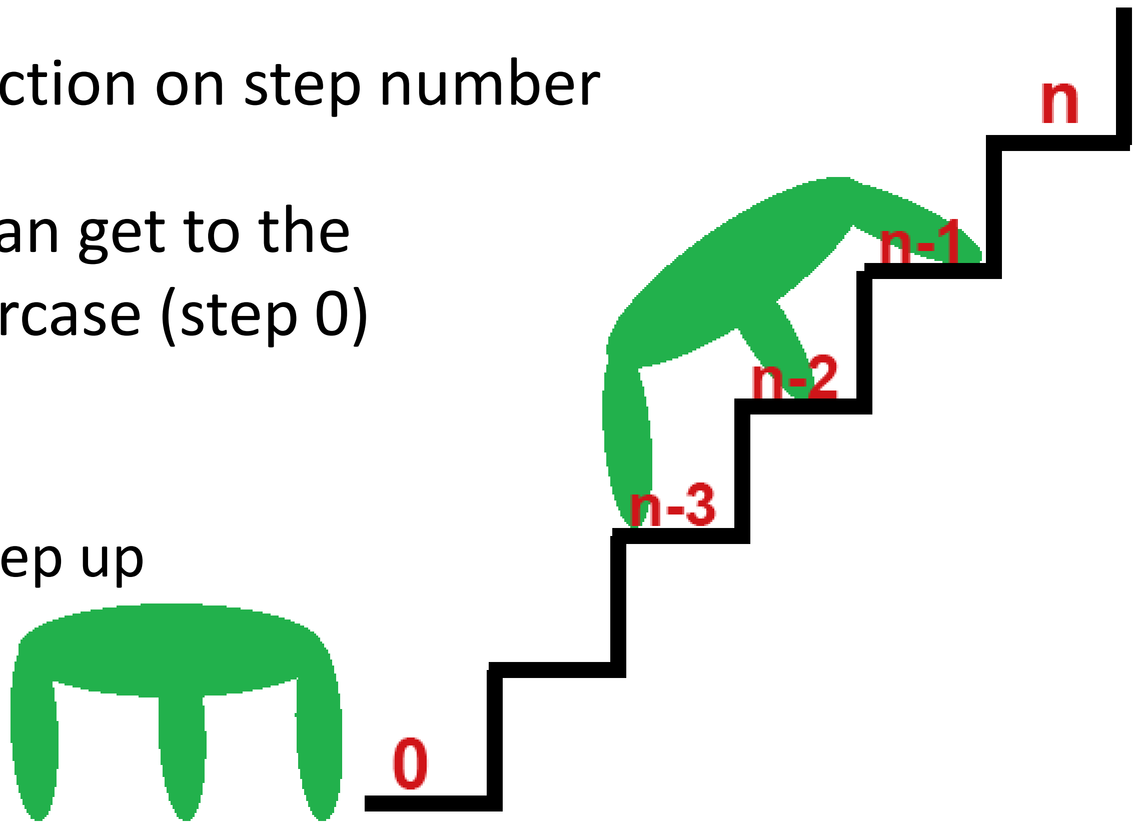
- Prove that robot R can go up the staircase to any arbitrary step

- Proof by simple induction on step number

- **Basis**: prove that R can get to the beginning of the staircase (step 0)

- **Induction step:**

Prove that R can take a step up
(If R can get to steps k<n,
it can go to step n)



Framework for proofs by induction on formulae

To prove P holds for any formula, take these steps:

- **Basis**: Prove P holds for atomic formula X (complexity=0)
- **Induction step:**
  - Assume P holds for all formula with complexity k<n, where n is complexity of X
  - Prove P holds for X
    - If X has the form $(\neg A)$
    - If X has the form (A o B), where o $\in\{\wedge, \vee, \rightarrow, \equiv\}$

A few Metatheorems

- **Theorem**. Every Boolean formula A has the same number of left and right brackets. (Proof by induction on formulae)
- **Corollary**. Any nonempty proper prefix of a Boolean formula A has more left than right brackets. (Proof by induction on formulae)
- **Theorem**. **(Unique Readability)** For any formula A, its immediate predecessors are uniquely determined.
  - Proof by contradiction- showing it is impossible to have different sets of i.p.s