MATH/CSE 1019 Third test
Fall 2011
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1. (15 points) Cardinality: Determine which of these sets is countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
   (a) (5 points) The odd negative integers
   Solution: This set (call it S_1) is countably infinite. The set of positive integers is also the set of natural numbers $\mathbb{N}$. The correspondence is $f : \mathbb{N} \to S_1, f(n) = -(2n-1)$.
   (b) (5 points) The integers that are multiples of 7.
   Solution: This set (call it S_2) contains both positive and negative integers. So let us use 0 to map to 0, the even numbers to map to the positive numbers and the odd numbers to map to the negative number. The correspondence is $f : \mathbb{N} \to S_2, f(0) = 0$ and for $k > 0$, $f(2k) = 7k, f(2k-1) = -7k$.
   (c) (5 points) The integers less than 100.The correspondence is $f : \mathbb{N} \to S_2, f(0) = 0$ and for $k > 0$, $f(2k) = 7k, f(2k-1) = -7k$.
   Solution: In this case we have to dispense with the positive integers first, before dealing with the negative ones. The correspondence is $f(n) = n$ if $1 \le n \le 99$ and for $n > 99$, $f(n) = 100 - n$.

2. (15 points) Induction:
   (a) (5 points) Prove using induction that $2^n > n^2$ if n is an integer greater than 4.
   Solution: For the base case we use $n = 5$. Then the statement says $2^5 - 32 > 5^2 = 25$ which is true. For the inductive step let the statement be true for $n = k$. Then $2^k > k^2$. For $n = k+1$ we have

$$
\begin{aligned}
LHS &= 2^{k+1} && (1)\\
&= 2 * 2^k && (2)\\
&> 2k^2 \text{ by the inductive hypothesis for } n = k && (3)\\
&> (k+1)^2 \text{ if } 2k^2 > (k+1)^2 && (4)
\end{aligned}
$$

   To prove that $2k^2 > (k+1)^2$, we see that

$$
\begin{aligned}
2k^2 - (k+1)^2 &= k^2 - 2k - 1 && (5)\\
&= (k-1)^2 - 2 && (6)\\
&> 0 \text{ for } k > 4 && (7)\\
& && (8)
\end{aligned}
$$

   (b) (5 points) Prove using induction that 3 divides $n^3 + 2n$ whenever n is a positive integer.
   Solution: For the base case we use $n = 1$. Then the statement says $1^3 + 2 * 1 = 3$ is divisible by 3, which is true.
   For the inductive step let the statement be true for $n = k$. Then $k^3 + 2k$ is divisible by 3. For $n = k+1$ we have

$$
\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 2k + 3k^2 + 3k + 1 + 2 && (9)\\
&= (k^3 + 2k) + 3(k^2 + k + 1) && (10)
\end{aligned}
$$

   The first bracketed expression is divisible by 3 by the inductive hypothesis. The second bracketed expression has a factor of 3 multiplied with a polynomial in k with integer coefficients, so that it too is divisible by 3.

(c) (5 points) Use strong induction to show that every positive integer can be written as a sum of distinct powers of 2, that is, as a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$ and so on.
Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. Where it is even, note that $\frac{k+1}{2}$ is an integer.

Solution:

Note: The word distinct is crucial, otherwise every number can be written as a sum of 1's.

For the base case we use $n = 1$. Then the statement is true because $1 = 2^0$.

For the inductive step let the statement be true for all $n \leq k$. Then for $n = k + 1$ we consider 2 cases.

Case 1: $k + 1$ is even. Then $k + 1 = 2p$ for some integer $p > 0$, and by the inductive hypothesis p can be written as a sum of distinct powers of 2. This implies that $2p$ is a sum of distinct powers of 2, because each distinct power of 2 in p is multiplied by 2.

Case 2: $k + 1$ is odd. So k is even. By the inductive hypothesis, k is a sum of distinct powers of 2 and further it cannot have 2^0 because every other power of 2 is even and thus the sum k would be odd. It follows that $k + 1$ can be expressed as a sum of distinct powers of 2 by adding $2^0 = 1$ to the sum of distinct powers of 2 that form k.

Thus in each case $k + 1$ can be expressed as a sum of distinct powers of 2.