MATH/CSE 1019 Final Examination
Fall 2011
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1. (2 points) Evaluate the infinite geometric series
$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$$
Note that the left hand side is $0.99\ldots$. Use the answer above to fill in the blank below
Solution: This is a geometric series with $a = 0.9$ and $r = 0.1$. So the infinite sum is $\frac{a}{1-r} = \frac{0.9}{1-0.1} = 1$.
$0.99\ldots = 1$.

2. (2 points) Express the following statement in predicate logic: "given any 2 distinct rational numbers, there exists a rational number between them in value".
Solution: Make the domain $\mathbb{Q}$, the set of rational numbers. Then, the given statement is
$$\forall x \forall y[(x \neq y) \rightarrow (\exists z(x < z < y) \vee (x > z > y))].$$

3. (2 points) Prove that if $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}, f(n) = n^2 - 100n$ then $f(n) \in \Omega(n^2)$.
Solution: Note that $n^2 - 100n \geq n^2/2$ if $n \geq 200$. So we use $n_0 = 200$, $c = 0.5$ in the definition of $\Omega(n^2)$ and thus $f(n) \in \Omega(n^2)$.

4. (2 points) What is the negation of $\forall x \exists y P(x) \rightarrow Q(y)$. Your answer should not have negation symbols before quantifiers and no implication symbols.
Solution: First we rewrite the given statement as $\forall x \exists y(\neg P(x) \vee Q(y))$. Then we can negate this in the usual way and get
$$\exists x \forall y(P(x) \wedge \neg Q(y))$$

5. (2 points) Write down the function that the following recursive algorithm computes and give a 1-sentence justification for your answer. Assume that n is a positive integer.
```
Mystery(n)
if n = 1
   then return 1
   else return (n + Mystery(n - 1))
```
Solution: The program computes $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$.
Each recursive call adds the previous number, stopping at 1, so it computes the sum $n+(n-1)+\ldots+2+1$.

6. (3 points) Prove using induction the inequality $n < 2^n$ for all $n >= 0, n \in \mathbb{Z}$.
Solution: We check that it is true for $n = 0, 1$ and then prove the rest using induction. It holds for $n = 0, 1$ because $0 < 2^0 = 1$ and $1 < 2^1 = 2$.
Base Case ($n = 2$): True because $2 < 2^2 = 4$.
Inductive Step: Assume the statement is true for $n = k$. So $k < 2^k$. Now for $n = k + 1$ we have
$$\begin{aligned} RHS &= 2^{k+1} \\ &= 2 * 2^k \\ &> 2k \\ &> k + 1 \text{ for all } k > 1. \end{aligned}$$

7. (1+2 points) Is $n \in \Omega(2^n)$ when n is a positive integer? Justify your answer.

   Solution: No.

   $f(n) = n$ is a linear function of n and $g(n) = 2^n$ is an exponential function of n. Exponential functions grow asymptotically faster than any polynomial functions, including linear functions.

8. (4 points) Consider the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 1$. Prove that f is a bijection.

   Solution:

   First we prove that f is 1-1.This is true, since if $f(x) = f(y)$ we have $2x + 1 = 2y + 1$ or $x = y$.

   Next we prove that f is onto. Consider any $y \in \mathbb{R}$. Choose $x = \frac{y-1}{2}$. Then $f(x) = y$.

   Since f is 1-1 and onto, it is a bijection.

9. (5 points) Prove using induction that $1 + 3 + \ldots + (2n - 1) = n^2$, where n is a positive integer.

   Solution: There is a subtlety here, we need to write the sum as $\sum_{m=1}^{n}(2m - 1) = n^2$ and do an induction on n.

   Base Case ($n = 1$): True because $1 = 1^2 = 1$.

   Inductive Step: Assume the statement is true for $n = k$. So $\sum_{m=1}^{k}(2m - 1) = k^2$. Now for $n = k + 1$ we have

$$\begin{aligned}\sum_{m=1}^{k+1}(2m-1) &= \sum_{m=1}^{k}(2m-1) + 2k + 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2\end{aligned}$$

10. (1+4 points) Consider the set of positive integers not divisible by 3. Is this set finite, countably infinite or uncountable? Prove your answer.

    Solution: The set is countable.

    One valid argument is that the set given is a subset of $\mathbb{N}$ which is countable.

    Another approach is to

    The set given is a set of all numbers of the form $3n \pm 1$. It can be expressed as the union of two sets, $\{x | x = 3n + 1, n \in \mathbb{N}\} \cup \{x | x = 3n - 1, n \in \mathbb{N}\}$. Each of these sets is countable since $3n - 1, 3n + 1$ are bijections from $\mathbb{N}$ to these sets. We can show the union to be countable by mapping all odd natural numbers to one set and all the even natural numbers to the other.

11. (4 points) Give a recursive definition of the sequence $\{a_n\} = 10^n, n = 1, 2, 3, \ldots$.

    Solution: The definition is as follows.

    $10 \in S$
    $\forall x \in S \forall y \in S, xy \in S$.

12. (6 points) Using loop invariants, prove that the following algorithm for exponentiation (computing y^z where $y \in \mathbb{R}, z \in \mathbb{N}$) is correct. If you like you can use the loop invariant: "before iteration i ($i = 1, 2, ..., z_0 + 1$ where z_0 is the initial value of z), $x = y^{i-1}$ and $z = z_0 - i + 1$".

    POWER(y, z)
    $x \leftarrow 1$
    while $z > 0$
    do $x \leftarrow x * y$
    $\quad z \leftarrow z - 1$
    return (x)

Solution:

Not in our syllabus.

13. (3+2 points) What is the value returned by the following function, when n is a positive integer? Give a 1-sentence justification. Express your answer as a function of n only.

MYSTERY2(n)
```
1  result ← 1
2  for i ← 1 to n
3  do for j ← i to n
4       do result ← result + 2
5  return (result)
```

What is the value returned if you replace the + in line 4 with $*$?

Solution: Update: The previous solution was incorrect because I read the inner loop as going from 1 to n. The corrected solution is as follows.

The program computes $n^2 + n + 1$.

Line 4 is executed $\sum_{i=1}^{n}(n - i + 1)$ times because it is inside two nested for loops.

$$\begin{aligned}\sum_{i=1}^{n}(n-i+1) &= \sum_{i=1}^{n} n - \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= n^2 - \frac{n(n+1)}{2} + n \\ &= \frac{2n^2 - n^2 - n + 2n}{2} \\ &= \frac{n^2+n}{2}\end{aligned}$$

Each time it executes it adds to the result, giving a total of $n^2 + n + 1$.

If line 4 multiplied the result by 2 each time it was executed, the value returned would be $2^{\frac{n^2+n}{2}}$.

14. (5 points) Using the binomial theorem, show that $C(n, k) < 2^n$ for all $n \in \mathbb{Z}^+$ and $0 \leq k \leq n$.

Solution: Not in our syllabus.

15. (5 points) Show that if 7 integers are selected from the first 10 positive integers (i.e., the numbers 1 thorugh 10), there must be at least 2 pairs of these integers with sum 11.

Solution: Note that there are 5 pairs that sum to $11 - 1 + 10, \ldots, 5 + 6$. Think of these pairs as bins and the 7 integers being chosen as the balls. Note also that each bin can have at most 2 balls (selections). So if there are 7 numbers, at least two bins will have two balls each. So there are at least 2 pairs of integers that sum to 11.

16. (5 points) How many ways are there for 10 women and 6 men to stand in a line so that no two men stand next to each other?
Hint: First position the women and then consider possible positions for the men.

Solution:

Not in our syllabus.