MATH/EECS 1019 Second test (version 1)
Fall 2014
Solutions
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1. (6 points) Sets

   (a) (1+2 points) Construct Venn diagrams for each of these combinations of the sets A, B, C.
   (i) $A \cup (B \cap C)$ (ii) $\overline{A} \cap \overline{B} \cap \overline{C}$
   Solution:

   (b) (3 points) Recall that the power set of a set A is the set of all subsets of A. Show that if A is a subset of B then the power set of A is a subset of the power set of B.
   Solution: Consider the power set $\mathcal{P}(A)$ of set A. Take any element $x \in \mathcal{P}(A)$. By the definition of $\mathcal{P}(A)$, $x \subseteq A$. Since $A \subseteq B$ therefore it follows that $x \subseteq B$. However, every subset of B is an element of the power set $\mathcal{P}(B)$ of B. So $x \in \mathcal{P}(B)$. Thus every element of $\mathcal{P}(A)$ is in $\mathcal{P}(B)$. This implies that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

2. (6 points) Proofs

   (a) (3 points) Suppose A, B, C are sets. Prove or disprove: $(A - B) - C = A - (B - C)$.
   Solution: We can disprove this with a counterexample. Let $A = \{1, 2, 3, 4, 5\}, B = \{2, 3, 5, 6\}$ and $C = \{4, 5, 6, 7\}$. Then
   $(A - B) - C = \{1, 4\} - \{4, 5, 6, 7\} = \{1\}$ and
   $A - (B - C) = \{1, 2, 3, 4, 5\} - \{2, 3\} = \{1, 4, 5\}$

   (b) (3 points) Prove that there is no positive integer n such that $n^2 + n^3 = 99$.
   Solution: We know that n^3 is an increasing function o n and that $5^3 = 125 > 99$. Therefore the only candidates for n that we need to consider are $n = 1, 2, 3, 4$. We can check that for each of these values of n the given equation is not satisfied. Therefore the assertion in the question is true.

3. (6 points) Proofs

   (a) (2 points) Prove that $\sqrt[3]{2}$ is an irrational number.
   Solution: We mimic the proof of the irrationality of $\sqrt{2}$. Suppose, for the sake of contradiction that $\sqrt[3]{2}$ is rational. So $\sqrt[3]{2} = p/q$ for some integers $p, q, q \neq 0$. Without loss of generality, we can assume that p, q have no common factors. Then we have,

$$\begin{aligned} p/q &= \sqrt[3]{2} \\ p^3/q^3 &= 2 \\ p^3 &= 2q^3 \end{aligned}$$

   So p^3 is even since it is equal to an even number, and so p is even (we proved this in class earlier)

$$\begin{aligned} p &= 2k \text{ for some integer } k \\ 2q^3 &= 8k^3 \\ q^3 &= 4k^3 \end{aligned}$$

   So q^3 is even since it is equal to an even number, and so q is even (we proved this in class earlier)

   If p, q are both even, they have a common factor 2, which is a contradiction. Thus $\sqrt[3]{2}$ is an irrational number.

(b) (4 points) Let $P(n)$ be the statement

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

where n is an integer greater than 1. Prove that this statement is true for $n \in \mathbb{N}, n \geq 2$ using mathematical induction.

Solution: We prove this by inducton on n.

Base Case: For $n = 1$, $1 = 2 - 1/1 = 1$. So the statement holds.

Inductive Step: Assume that the hypothesis holds for $n = k$, i.e., $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} < 2 - \frac{1}{k}$. Then for $n = k + 1$, we have

$$\begin{aligned}
1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &= [1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2}] + \frac{1}{(k+1)^2} \\
&< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&< 2 - \frac{1}{k+1} \text{ because} \\
\frac{1}{k} - \frac{1}{k+1} &= \frac{1}{k(k+1)} \\
&> \frac{1}{(k+1)^2}
\end{aligned}$$

Thus if the hypothesis holds for $n = k$ it also holds for $n = k + 1$. Therefore, the statement given in the question is true.

4. (6 points) Proofs

(a) (3 points) Prove that if n is a nonnegative integer then $n^3 - n$ is divisible by 6.

Solution: We could prove this by induction but a direct proof is easier.

Note that $n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$. This is a product of 3 consecutive integers. Note that in any set of 3 consecutive integers, there must be at least 1 even number and at least 1 multiple of 3. Thus the product is divisible by 6.

(b) (3 points) Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

Solution: We prove this by inducton on n.

Base Case: For $n = 1$, $4^{n+1} + 5^{2n-1} = 4^2 + 5^1 = 21$ which is divisible by 21.

Inductive Step: Assume that the hypothesis holds for $n = k$, i.e., $4^{k+1} + 5^{2k-1} = 21A$ for some integer A. Then for $n = k + 1$,

$$\begin{aligned}
4^{n+1} + 5^{2n-1} &= 4^{k+1+1} + 5^{2(k+1)-1} \\
&= 4^{k+2} + 5^{2k+1} \\
&= 4 * 4^{k+1} + 25 * 5^{2k-1} \\
&= 4 * 4^{k+1} + 4 * 5^{2k-1} + 21 * 5^{2k-1} \\
&= 4[4^{k+1} + 5^{2k-1}] + 21 * 5^{2k-1} \\
&= 4 * 21A + 21 * 5^{2k-1} \\
&= 21[4A + 5^{2k-1}]
\end{aligned}$$

Thus if the hypothesis holds for $n = k$ it also holds for $n = k + 1$. Therefore, the statement given in the question is true.

5. (6 points) Proofs

   (a) (3 points) Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. Use induction or strong induction to prove that

$$a_1 + a_2 + \ldots + a_n \geq n$$

Solution: This is a very difficult problem. The solution is as follows. We will use induction on n.

Base Case: For $n = 1$, we are given that $a_1 = 1$ which satisfies $a_1 \geq 1$.

Inductive Step: Assume that the hypothesis holds for $n = k > 2$. So for any positive real numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 a_2 \ldots a_k = 1$, it must be the case that $a_1 + a_2 + \ldots + a_k \geq k$. Now consider any positive real numbers $a_1, a_2, \ldots, a_k, a_{k+1}$ satisfying $a_1 a_2 \ldots a_k a_{k+1} = 1$. Let us assume that no $a_i = 1$, because we can leave out such a_i and be left to prove the same statement for smaller n. So all the a_i's are either less than 1 or greater than 1. Since the product is 1, there must be at least one a_i that is less than 1, and at least one a_i that is less than 1. Without loss of generality, call these two a_k, a_{k+1}. Let $b = a_k a_{k+1}$. Then the set of k numbers $a_1, a_2, \ldots, a_{k-1}, b$ satisfies the conditions of the inductive hypothesis ($a_1 a_2 \ldots a_{k-1} b = 1$) and thus we conclude that

$$a_1 + a_2 + \ldots + a_{k-1} + b \geq k,$$

or

$$a_1 + a_2 + \ldots + a_{k-1} \geq k - b.$$

From here we see that if we can show $a_k + a_{k+1} \geq 1 + b$, we are done because then we would add $a_k + a_{k+1}$ to both sides of the last inequality and get

$$\begin{aligned} a_1 + a_2 + \ldots + a_{k-1} + a_k + a_{k+1} &\geq k - b + 1 + b \\ &= k + 1 \end{aligned}$$

and that would give us what we want to prove. Now note that $(1 - a_k)(1 - a_{k+1}) \leq 0$ because one of the terms in the product is positive and the other is negative.

$$\begin{aligned} (1 - a_k)(1 - a_{k+1}) &\leq 0 \\ 1 - a_k - a_{k+1} + a_k a_{k+1} &\leq 0 \\ 1 - a_k - a_{k+1} + b &\leq 0 \\ 1 + b \leq a_k + a_{k+1} & \end{aligned}$$

   (b) (3 points) Prove that at any party with at least two people, there must be two people who know the same number of other people there.

Solution: We can make a simplifying assumption here – that every person knows at least one other person at the party. If this is the case then each of n people know at least 1 and at most $n - 1$ other people. By the Pigeonhole Principle, there must be two people who know the same number of people at the party.

Note:

1. This is not correct, but I gave you full marks if you made this argument.
2. You do not need to assume that knowing is mutual (or symmetric).

The correct solution considers the case where one or more persons may know zero other people at the party. If there are two or more people who know no other people at the party we are done. If there is exactly one person who knows no other people, we leave him/her out and then we get a problem where our simplifying assumption above holds, and thus the proof above is now correct.