MATH/CSE 1019 First test - sample
Solutions
Instructor: S. Datta

1. (6 points) Propositional Logic

   (a) (3 points) Write down the truth table for the following proposition.

$$(p \wedge q) \vee \neg r$$

Solution:

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

   (b) (3 points) State the converse, contrapositive and inverse of the following statement: "A positive integer is a prime only if it has no divisors other than 1 and itself".

Solution: As discussed in class, this is the equivalent to "If a positive integer is prime then it has no divisors other than 1 and itself".

Contrapositive: If a positive integer has a divisor that is not 1 or itself, then it is not prime.

Inverse: If a positive integer is not prime then it has divisors other than 1 and itself.

Note: We know that the contrapositive and inverse are not equivalent in general. So why are they equivalent in this example? The reason is that the statement is actually a bidirectional implication but only one direction is stated in the given sentence.

2. (6 points) Propositional equivalences

   (a) (3 points) Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent.

Solution: Use a truth table.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$	$(p \wedge q)$	$(p \wedge q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	F	T
T	F	F	F	T	T	F	T
F	T	T	T	T	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

   (b) (3 points) Show that $(p \wedge q) \to p$ is a tautology.

Solution: Use a truth table.

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

3. (6 points) Predicates

   (a) (3 points) Translate in 2 ways the following statement into a logical expression using predicates, quantifiers and logical connectives. First let the domain be the students in your class and second, let it consist of all people: "There is a person in your class who cannot swim".

   Solution: Define for the domain of all people the predicate $S(x)$ that is true is x can swim and false otherwise. Since it is defined for all people, it is also defined over the smaller domain of your classmates. So the given statement is $\exists x \neg S(x)$.

   Define also another predicate $C(x)$ that is true if x in in your class and false otherwise.

   Then the given statement is $\exists x C(x) \wedge \neg S(x)$.

   (b) (3 points) Express the negation of this proposition using quantifiers and then express the negation in English: "Some drivers do not obey the speed limit".

   Solution: Define for the domain of people who are drivers the predicate $O(x)$ that is true if x obeys the speed limits and false otherwise. Then the given statement is
   $\exists x \neg O(x)$.
   Its negation is $\forall x O(x)$.
   The equivalent English sentence is "All drivers obey the speed limit".

4. (6 points) Nested quantifiers A Discrete Math class contains 1 math major who is a freshman, 12 math majors who are spohomores, 15 CSE majors who are sophomores, 2 Math majors who are juniors, 2 CSE majors who are juniors, and 1 CSE major who is a senior. Express each of the following statements in terms of quantifiers and say why it is true or false.

   (a) (3 points) There is a student in the class who is neither a math major nor a junior.

   Solution: Let the domain of persons be students in this class, the domain of majors is the set $M = \{Math, CSE\}$ and the domain of year be the set $Y = \{Freshman, Sophomore, Junior, Senior\}$. Now define a predicate $In(x, m, y)$ to be true if x is in major m and year y.

   Note that if a student is not a math major they must be a CSE major. So the given statement is $\exists x In(x, CSE, Freshman) \vee In(x, CSE, Sophomore) \vee In(x, CSE, Senior)$.

   This is true, because there are many CSE majors who are not juniors.

   (b) (3 points) There is a major such that there is a student in the class in every year of study with that major.

   Solution: Then the given statement is $\exists m \in M \forall y \in Y \exists x In(x, m, y)$.

   This is false because CSE majors contain no freshmen an Math majors contain no seniors.

5. (6 points) Inference

   (a) (3 points) Explain what rule of inference are used to infer the following: "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre.

   Solution: For the domain of all people, define the predicates $F(x)$ that is true if x has visited France and false otherwise. Similarly define $L(x)$ to be true if x has visited Louvre and false otherwise. Finally consider the restricted domain of this class.

   So the given statements can be written as:

   $\exists x F(x)$
   $\forall x F(x) \rightarrow L(x)$
   $\therefore \exists x L(x)$

   This inference is valid because from $\exists x F(x)$ we can conclude $F(a)$ for some a by existential instantiation. Then by universal instantiation we get $F(a) \rightarrow L(a)$. Then by modus ponens we get $L(a)$. Finally by existential generalization we get $\exists x L(x)$.

(b) (3 points) Determine whether this is valid: "If x is a positive real number, then x^2 is a positive real number. Therefore if a^2 is positive, then a is a positive real number.

Solution: Let the domain be real numbers. Let $Positive(x)$ be the predicate that is true if x is positive and false otherwise. Then the given statements can be written as:

$\forall x Positive(x) \to Positive(x^2)$
$\therefore Positive(a^2) \to Positive(a)$

We can infer from $\forall x Positive(x) \to Positive(x^2)$, the statement $Positive(a) \to Positive(a^2)$ using universal instantiation. From there we cannot infer the converse. So the given inference is invalid.