Counting the number of elements

- What is counting?
 - Labeling with integers
 - Correspondence with integers

Cardinality

A set S has k elements if and only if there exists a bijection between S and {1,2,...,k}.

S and {1,...,k} have the same <u>cardinality</u>.

If there is a surjection possible from $\{1,...,n\}$ to S, then $n \ge |S|$.

We can generalize this way of comparing the sizes of sets to infinite ones.

Countably Infinite Sets

A set S is <u>infinite</u> if there exists a surjective function $F:S \rightarrow N$.

"The set N has no more elements than S."

A set S is <u>countable</u> if there exists a surjective function F: $N \rightarrow S$ "The set S has not more elements than N."

A set S is <u>countably infinite</u> if there exists a bijective function F: $N \rightarrow S$. "The sets N and S are of equal size."

Counterintuitive facts

- Cardinality of even integers
 - Bijection i \leftrightarrow 2i
 - A proper subset of N has the same cardinality as N !
 - Same holds for odd integers
- What about pairs of natural numbers?

– Bijection from N to N x N !!

- Cantor's idea: count by diagonals
- Implies set of rational numbers is countable

Counterintuitive facts - 2

- Note that the ordering of Q is not in increasing order or decreasing order of value.
- In proofs, you CANNOT assume that an ordering has to be in increasing or decreasing order.
- So cannot use ideas like "between any two real numbers x, y, there exists a real number 0.5(x+y)" to prove uncountability.

More Countably Infinite Sets

One can make bijections between N and Integers (Z):

1 2 3 4 5 6 7 8 9 10 11 0 +1 -1 +2 -2 +3 -3 +4 -4 +5 -5

Summary

A set S is <u>countably infinite</u> if there exists a bijection between $\{0, 1, 2, ...\}$ and S.

Intuitively: A set S is countable, if you can make a List (numbering) $s_1, s_2, ...$ of all the elements of S.

Q: Are there bigger sets?

Uncountable Sets

There are infinite sets that are not countable. Typical examples are R, P(N)

We prove this by a <u>diagonalization argument</u>. In short, if S is countable, then you can make a list $s_1, s_2, ...$ of all elements of S.

Diagonalization shows that given such a list, there will always be an element x of S that does not occur in $s_1, s_2, ...$

Uncountability of **P** (N)

The set P (N) contains all the subsets of $\{1,2,...\}$. Each subset X \subseteq N can be identified by an infinite string of bits $x_1x_2...$ such that $x_j=1$ iff $j \in X$.

There is a bijection between P(N) and $\{0,1\}^N$.

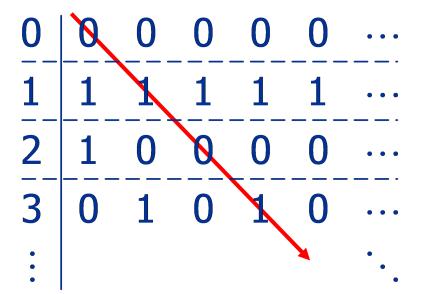
Proof by contradiction: Assume P (N) countable. Hence there must exist a surjection F from N to the set of infinite bit strings. "There is a list of *all* infinite bit strings."

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Diagonalization

Try to list all possible infinite bit strings:



Look at the bit string on the diagonal of this table: 0101... The negation of this string ("1010...") does not appear in the table.

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No Surjection $\mathbb{N} \rightarrow \{0,1\}^{\mathbb{N}}$

Let F be a function $N \rightarrow \{0,1\}^N$. F(1),F(2),... are all infinite bit strings.

Define the infinite string $Y=Y_1Y_2...$ by $Y_j = NOT(j-th bit of F(j))$

On the one hand $Y \in \{0,1\}^N$, but on the other hand: for every $j \in N$ we know that $F(j) \neq Y$ because F(j) and Y differ in the j-th bit.

F cannot be a surjection: $\{0,1\}^{N}$ is uncountable.

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Generalization

- We proved that the set of all binary strings is uncountably infinite.
- Can be generalized to set of strings over any base.

R is uncountable

- Similar diagonalization proof. We will prove [0,1) uncountable
- Let F be a function N → R
 F(1),F(2),... are all infinite digit strings (padded with zeroes if required).
- Define the infinite string of digits $Y=Y_1Y_2...$ by $Y_i = F(i)_i + 1$ if $F(i)_i < 8$ 7 if $F(i)_i \ge 8$
- Q: Where does this proof fail on N?

Other infinities

- We proved 2^N uncountable. We can show that this set has the same cardinality as P (N) and R.
- What if we take P (R)?
- Can we build bigger and bigger infinities this way?
- Cantor: Continuum hypothesis YES!