Counting the number of elements

- What is counting?
  - Labeling with integers
  - Correspondence with integers

Cardinality

A set S has k elements if and only if there exists a bijection between S and $\{1,2,\ldots,k\}$.

S and $\{1,\ldots,k\}$ have the same cardinality.

If there is a surjection possible from $\{1,\ldots,n\}$ to S, then $n \geq |S|$.

We can generalize this way of comparing the sizes of sets to infinite ones.

Countably Infinite Sets

A set S is <u>infinite</u> if there exists a surjective function $F: S \to \mathbb{N}$.
"The set $\mathbb{N}$ has no more elements than S."

A set S is <u>countable</u> if there exists a surjective function $F: \mathbb{N} \to S$
"The set S has not more elements than N."

A set S is <u>countably infinite</u> if there exists a bijective function $F: \mathbb{N} \to S$.
"The sets $\mathbb{N}$ and S are of equal size."

Counterintuitive facts

- Cardinality of even integers
  - Bijection i $\leftrightarrow$ 2i
  - A proper subset of N has the same cardinality as N !
  - Same holds for odd integers
- What about pairs of natural numbers?
  - Bijection from N to N x N !!
  - Cantor's idea: count by diagonals
  - Implies set of rational numbers is countable

Counterintuitive facts - 2

- Note that the ordering of Q is not in increasing order or decreasing order of value.
- In proofs, you CANNOT assume that an ordering has to be in increasing or decreasing order.
- So cannot use ideas like “between any two real numbers x, y, there exists a real number 0.5(x+y)” to prove uncountability.

More Countably Infinite Sets

One can make bijections between $\mathbb{N}$ and Integers (Z):

1	2	3	4	5	6	7	8	9	10	11
0	+1	-1	+2	-2	+3	-3	+4	-4	+5	-5

Summary

A set S is <u>countably infinite</u> if there exists a bijection between {0,1,2,…} and S.

Intuitively: A set S is countable, if you can make a List (numbering) $s_1, s_2, \ldots$ of all the elements of S.

Q: Are there bigger sets?

Uncountable Sets

There are infinite sets that are not countable.
Typical examples are R, P (N)

We prove this by a diagonalization argument.
In short, if S is countable, then you can make a list $s_1, s_2, \ldots$ of all elements of S.

Diagonalization shows that given such a list, there will always be an element x of S that does not occur in $s_1, s_2, \ldots$

Uncountability of $\mathcal{P}$ (N)

The set $\mathcal{P}$ (N) contains all the subsets of {1,2,…}.
Each subset $X \subseteq N$ can be identified by an infinite string of bits $x_1 x_2 \ldots$ such that $x_j = 1$ iff $j \in X$.

There is a bijection between $\mathcal{P}$ (N) and $\{0,1\}^N$.

Proof by contradiction: Assume $\mathcal{P}$ (N) countable.
Hence there must exist a surjection F from N to the set of infinite bit strings.
"There is a list of *all* infinite bit strings."

Diagonalization

Try to list all possible infinite bit strings:

0	0	0	0	0	0	…
1	1	1	1	1	1	…
2	1	0	0	0	0	…
3	0	1	0	1	0	…
⋮						⋱

Look at the bit string on the diagonal of this table: 0101… The negation of this string (“1010…”) does not appear in the table.

No Surjection $\mathbb{N} \rightarrow \{0,1\}^{\mathbb{N}}$

Let F be a function $N \rightarrow \{0,1\}^N$.
F(1),F(2),… are all infinite bit strings.

Define the infinite string $Y=Y_1Y_2\ldots$ by
Y_j = **NOT**(j-th bit of F(j))

On the one hand $Y \in \{0,1\}^N$, but on the other hand: for every $j \in N$ we know that $F(j) \neq Y$ because F(j) and Y differ in the j-th bit.

F cannot be a surjection: $\{0,1\}^N$ is uncountable.

Generalization

- We proved that the set of all binary strings is uncountably infinite.
- Can be generalized to set of strings over any base.

R is uncountable

- Similar diagonalization proof. We will prove [0,1) uncountable
- Let F be a function $N \to R$
  F(1),F(2),… are all infinite digit strings (padded with zeroes if required).
- Define the infinite string of digits $Y=Y_1Y_2\ldots$ by
  $Y_i = \mathbf{F(i)}_i + 1$ **if** $\mathbf{F(i)}_i < 8$
  7 **if** $\mathbf{F(i)}_i \geq 8$

Q: Where does this proof fail on N?

Other infinities

- We proved 2^N uncountable. We can show that this set has the same cardinality as P (N) and R.
- What if we take P (R)?
- Can we build bigger and bigger infinities this way?
- Cantor: Continuum hypothesis – YES!