Sets

- Unordered collection of elements, e.g.,
  - Single digit integers
  - Nonnegative integers
  - faces of a die
  - sides of a coin
  - students enrolled in 1019N, W 2007.
- Equality of sets
- Note: Connection with data types

Describing sets

- English description
- Set builder notation

Note:

The elements of a set can be sets, pairs of elements, pairs of pairs, triples, …!!

Cartesian product:

$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

Sets of numbers

- Natural numbers
- Whole numbers
- Integers
- Rational numbers
- Real numbers
- Complex numbers
- Co-ordinates on the plane

Sets - continued

- Cardinality – number of (distinct) elements
- Finite set – cardinality some finite integer n
- Infinite set - a set that is not finite

Special sets

- Universal set
- Empty set ϕ (cardinality = ?)

Sets vs Sets of sets

- {1,2} vs {{1,},{2}}
- {} vs {{}} = {ϕ}

Subsets

- $A \subseteq B$: $\forall x (x \in A \rightarrow x \in B)$

  Theorem: For any set S, $\phi \subseteq S$ and $S \subseteq S$.
- Proper subset: $A \subset B$: $\forall x (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$
- Power set P(S) : set of all subsets of S.
- P(S) includes S, ϕ.
- Tricky question – What is P(ϕ) ?

P(ϕ) = {ϕ}
Similarly, P({ϕ}) = {ϕ, {ϕ}}

Set operations

- Union – $A \cup B = \{ x \mid (x \in A) \vee (x \in B)\}$
- Intersection - $A \cap B = \{ x \mid (x \in A) \wedge (x \in B)\}$

  Disjoint sets - A, B are disjoint iff $A \cap B = \phi$
- Difference – $A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$

  Symmetric difference
- Complement – A^c or $\bar{A} = \{x \mid x \notin A\} = U - A$
- Venn diagrams

Laws of set operations

- Page 130 – notice the similarities with the laws for Boolean operators
- Remember De Morgan's Laws and distributive laws.
- Proofs can be done with Venn diagrams.

  E.g.: $(A \cap B)^c = A^c \cup B^c$

Proofs via membership tables (page 131)

Cartesian products

- A x B

Introduction to functions

A function from A to B is an assignment of exactly one element of B to each element of A.

E.g.:

- Let A = B = integers, f(x) = x+10
- Let A = B = integers, f(x) = x^2

Not a function

- A = B = real numbers f(x) = $\sqrt{x}$
- A = B = real numbers, f(x) = 1/x

Terminology

- A = Domain, B = Co-domain
- f: A $\rightarrow$ B (not "implies")
- range(f) = $\{y \mid \exists\, x \in A\ f(x) = y\} \subseteq B$
- int floor (float real){ … }
- $f_1 + f_2$, $f_1 f_2$
- One-to-one INJECTIVE
- Onto SURJECTIVE
- One-to-one correspondence BIJECTIVE

Operations with functions

- Inverse $f^{-1}(x) \neq 1/f(x)$
  $f^{-1}(y) = x$ iff $f(x) = y$
- Composition: If f: A $\rightarrow$ B, g: C $\rightarrow$ A, then $f \circ g$: C $\rightarrow$ B, $f{\circ}g(x) = f(g(x))$

Graphs of functions

Special functions

- All domains: identity $\Im(x)$
  Note: $f \circ f^{-1} = f^{-1} \circ f = \Im$

- Integers: floor, ceiling, DecimalToBinary, BinaryToDecimal
- Reals: exponential, log

Special functions

- DecimalToBinary, BinaryToDecimal
- E.g. $7 = 111_2$, $1001_2 = 9$
- BinaryToDecimal – $n = 1001_2$:
- $n = 1*2^3 + 0*2^2 + 0*2^1 + 1*2^0 = 9$

- DecimalToBinary – n = 7:
- b_1 = n rem 2 = 1, n = n div 2 = 3
- b_2 = n rem 2 = 1, n = n div 2 = 1
- b_3 = n rem 2 = 1, n = n div 2 = 0.
- STOP

Special functions – contd.

- Changing bases: In general need to go through the decimal representation
- E.g: $101_7 = ?_9$
- $101_7 = 1*7^2 + 0*7^1 + 1*7^0 = 50$
- Decimal to Base 9:
- d_1 = n rem 9 = 5, n = n div 9 = 5
- b_2 = n rem 9 = 5, n = n div 9 = 0.
- STOP
- So $101_7 = 55_9$

Special functions – tricks

- Changing bases that are powers of 2:
- Can often use shortcuts.
- Binary to Octal:
- 10111101 = 275_8

- Binary to Hexadecimal:
- 10111101 = BD_{16}

- Hexadecimal to Octal: Go through binary, not decimal.

Sequences

- Finite or infinite
- Calculus – limits of infinite sequences (proving existence, evaluation…)
- E.g.
  - Arithmetic progression (series)
    1, 4, 7, 10, …
  - Geometric progression (series)
    3, 6, 12, 24, 48 …

Similarity with series

- $S = a_1 + a_2 + a_3 + a_4 + \ldots..$ (n terms)
- Consider the sequence

  $S_1, S_2, S_3, \ldots S_n$, where

  $S_i = a_1 + a_2 + \ldots + a_i$

In general we would like to evaluate sums of series – useful in algorithm analysis.

e.g. what is the total time spent in a nested loop?

Sums of common series

- Arithmetic series

  e.g. $1 + 2 + \ldots + n$ (occurs in the analysis of running time of simple for loops)

  general form $\Sigma_i t_i$, $t_i = a + ib$
- Geometric series

  e.g. $1 + 2 + 2^2 + 2^3 + \ldots + 2^n$

  general form $\Sigma_i t_i$, $t_i = ar^i$
- More general series (not either of the above)

  $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2$

Sums of common series - 2

- Technique for summing arithmetic series
- Technique for summing geometric series
- More general series – more difficult

Caveats

- Need to be very careful with infinite series
- In general, tools from calculus are needed to know whether an infinite series sum exists.
- There are instances where the infinite series sum is much easier to compute and manipulate, e.g. geometric series with $r < 1$.

Cardinality revisited

- A set is finite (has finite cardinality) if its cardinality is some (finite) integer n.
- Two sets A,B have the same cardinality iff there is a one-to-one correspondence from A to B
- E.g. alphabet (lower case)
- a b c …..
- 1 2 3 …..

Infinite sets

- Why do we care?
- Cardinality of infinite sets
- Do all infinite sets have the same cardinality?

Countable sets

Defn: Is finite OR has the same cardinality as the positive integers.

- Why do we care?

E.g.

- – The algorithm works for "any n"
- – Induction!

Countable sets – contd.

- Proving this involves (usually) constructing an explicit bijection with positive integers.
- Fact (Will not prove): Any subset of a countable set is countable.

Will prove that

- The rationals are countable!
- The reals are not countable

The integers are countable

- Write them as

  0, 1, -1, 2, -2, 3, -3, 4, -4, ……
- Find a bijection between this sequence

  and 1,2,3,4,…..

  Notice the pattern:

  $1 \to 0$ $\quad 2 \to 1$ $\quad$ So f(n) = n/2 if n even

  $3 \to -1$ $\quad 4 \to 2$ $\quad\quad$ -(n-1)/2 o.w.

  $5 \to -2$ $\quad 6 \to 3$

Other simple bijections

- Odd positive integers

  $1 \to 1 \quad 2 \to 3 \quad 3 \to 5 \quad 4 \to 7$ ….
- Union of two countable sets A, B is countable:

  Say $f: N \to A$, $g:N \to B$ are bijections

  New bijection $h: N \to A \cup B$

  h(n) = f(n/2) if n is even

  = g((n-1)/2) if n is odd.

The rationals are countable

- Show that $Z^+ \times Z^+$ is countable.
- Trivial injection between Q^+, $Z^+ \times Z^+$.
- To go from Q^+ to Q, use the trick used to construct a bijection from Z to Z^+.
- Details on the board.

The reals are not countable

- Wrong proof strategy:
- Suppose it is countable
- Write them down in increasing order
- Prove that there is a real number between any two successive reals.

- WHY is this incorrect?

(Note that the above "proof" would show that the rationals are not countable!!)

The reals are not countable - 2

- Cantor diagonalization argument (1879)
- VERY powerful, important technique.
- Proof by contradiction.
- Sketch (details done on the board)
  - Assume countable
  - look at all numbers in the interval [0,1)
  - list them in ANY order
  - show that there is some number not listed

Notes

- The cardinality of neither the reals nor the integers are finite, yet one set is countable, the other is not.
- Q: Is there a set whose cardinality is “in-between”?
- Q: Is the cardinality of R the same as that of [0,1) ?