Mathematical Induction

- Very simple
- Very powerful proof technique
- “Guess” and verify strategy

Basic steps

- Hypothesis: P(n) is true for all positive integers n
- Base case/basis step (starting value)
- Inductive step

Formally:

$$[P(1) \land \forall k\,(P(k) \rightarrow P(k+1))] \rightarrow \forall n\, P(n)$$

Intuition

Iterative modus ponens:

$P(k)$

$P(k) \rightarrow P(k+1)$

$P(k+1)$

Need a starting point (Base case)

Proof is beyond the scope of this course

Example 1

P(n): $1 + 2 + \ldots + n = n(n+1)/2$

Follow the steps:

- Base case: P(1).

  LHS = 1. RHS = $1(1+1)/2$ = LHS

- Inductive step:
  - Assume P(n) is true.
  - Show P(n+1) is true.

Note: $1 + 2 + \ldots + n + (n+1)$

$= n(n+1)/2 + (n+1) = (n+1)(n+2)/2$ **done**

Example 2

- A difficult series (suppose we guess the answer)
- $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$
- Base case: P(1) LHS = 1 = RHS.
- Inductive step:

$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 =$
$n(n+1)(2n+1)/6 + (n+1)^2 =$
$(n+1)(n+2)(2n+3)/6 = \text{RHS}.$

Proving Inequalities

- P(n): $n < 4^n$
- Base case: P(1) holds since 1 < 4.
- Inductive step:
- Assume $n < 4^n$
- Show that $n+1 < 4^{n+1}$

$n+1 < 4^n + 1 < 4^n + 4^n < 4.4^n = 4^{n+1}$

More examples

- Sum of odd integers
- n^3-n is divisible by 3
- Number of subsets of a finite set

Points to remember

- Base case does not have to be n=1
- Most common mistakes are in not verifying that the base case holds

- Sometimes we need more than P(n) to prove P(n+1) – in these cases STRONG induction is used

- Usually guessing the solution is done first.

How can you guess a solution?

- Try simple tricks: e.g. for sums with similar terms – n times the average or n times the maximum; for sums with fast increasing/decreasing terms, some multiple of the maximum term.
- Often proving upper and lower bounds separately helps.

Strong Induction

- Equivalent to induction – use whichever is convenient
- Formally:

$[\ P(1) \wedge \forall k\ (P(1) \wedge \ldots \wedge P(k) \rightarrow P(k+1))\ ] \rightarrow \forall n\ P(n)$

- Often useful for proving facts about algorithms

Examples

- Fundamental Theorem of Arithmetic: every positive integer n, n >1, can be expressed as the product of one or more prime numbers.
- every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Fallacies/caveats

- "Proof" that all Canadians are of the same age!
  http://www.math.toronto.edu/mathnet/falseProofs/sameAge.html