Recursion (Part 2)

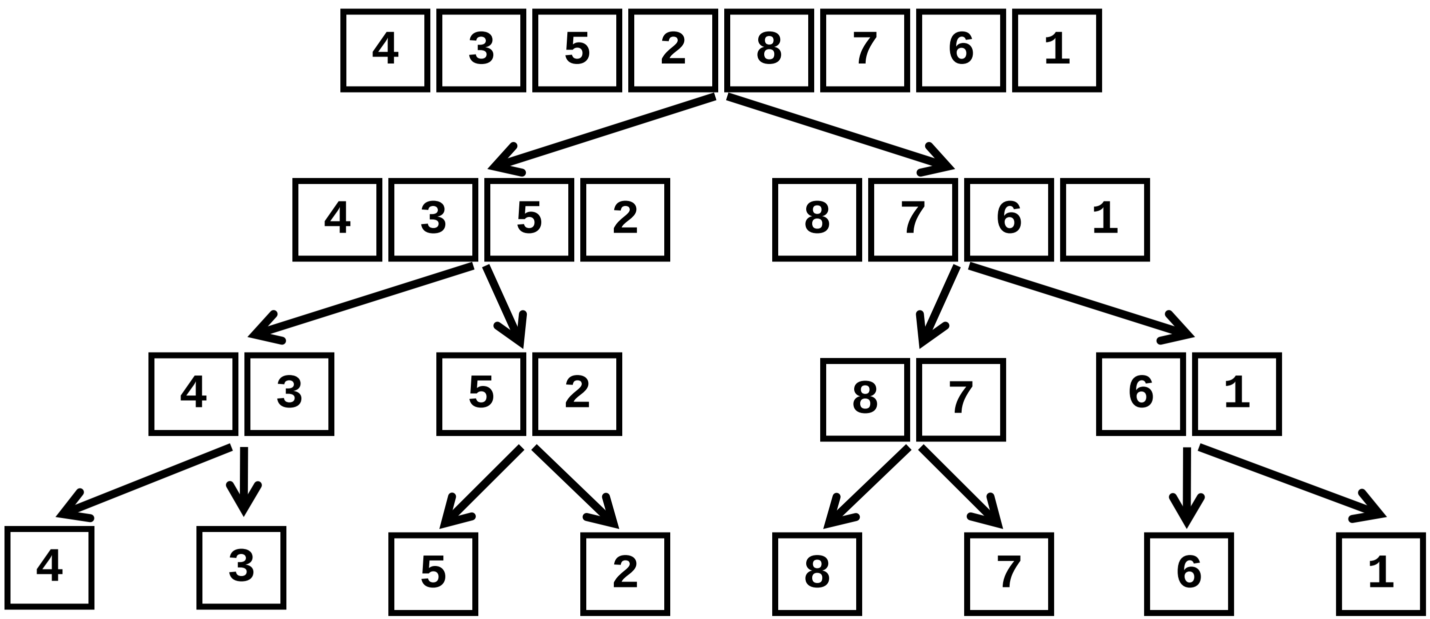
Solving Recurrence Relations

Divide and Conquer

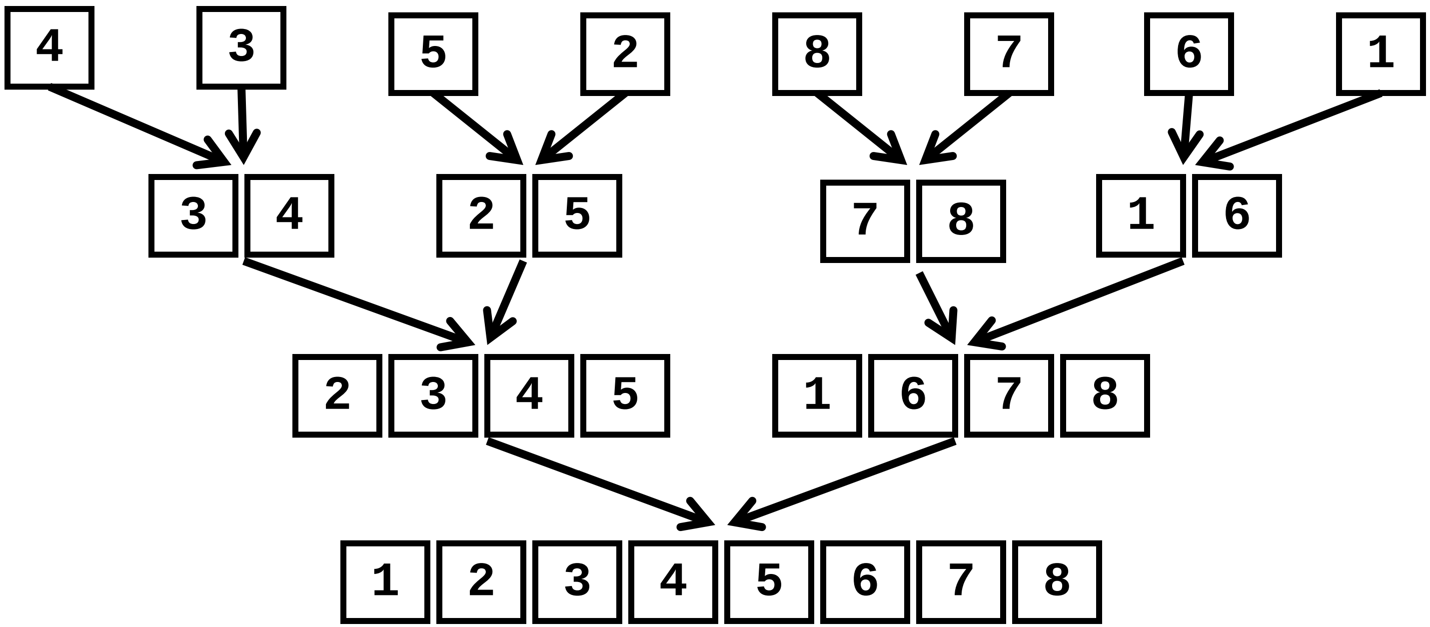
- bisection works by recursively finding which half of the range `'plus'` – `'minus'` the root lies in
  - each recursive call solves the same problem (tries to find the root of the function by guessing at the midpoint of the range)
  - each recursive call solves *one* smaller problem because half of the range is discarded
    - bisection method is decrease and conquer
- divide and conquer algorithms typically recursively divide a problem into several smaller sub-problems until the sub-problems are small enough that they can be solved directly

Merge Sort

- merge sort is a divide and conquer algorithm that sorts a list of numbers by recursively splitting the list into two halves

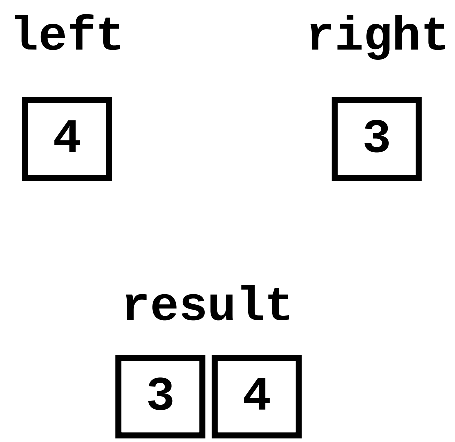


- the split lists are then merged into sorted sub-lists



Merging Sorted Sub-lists

- two sub-lists of length 1



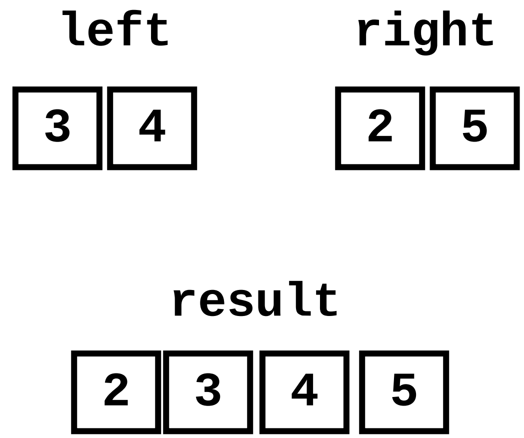
1 comparison
2 copies

```
LinkedList<Integer> result = new LinkedList<Integer>();

int fL = left.getFirst();
int fR = right.getFirst();
if (fL < fR) {
  result.add(fL);
  left.removeFirst();
}
else {
  result.add(fR);
  right.removeFirst();
}
if (left.isEmpty()) {
  result.addAll(right);
}
else {
  result.addAll(left);
}
```

Merging Sorted Sub-lists

- two sub-lists of length 2



3 comparisons
4 copies

```java
LinkedList<Integer> result = new LinkedList<Integer>();

while (left.size() > 0 && right.size() > 0 ) {
  int fL = left.getFirst();
  int fR = right.getFirst();
  if (fL < fR) {
    result.add(fL);
    left.removeFirst();
  }
  else {
    result.add(fR);
    right.removeFirst();
  }
}
if (left.isEmpty()) {
  result.addAll(right);
}
else {
  result.addAll(left);
}
```

Merging Sorted Sub-lists

- two sub-lists of length 4

left

2	3	4	5

right

1	6	7	8

result

1	2	3	4	5	6	7	8

5 comparisons
8 copies

Simplified Complexity Analysis

- in the worst case merging a total of **`n`** elements requires

  `n – 1` comparisons **`+`**

  `n` copies

  `= 2n – 1` total operations

- we say that the worst-case complexity of merging is the order of $O(n)$
  - $O(\ldots)$ is called Big O notation
  - notice that we don't care about the constants 2 and 1

- formally, a function $f(n)$ is an element of $O(n)$ if and only if there is a positive real number M and a real number m such that

$$|\, f(n) \,| < Mn \quad \text{for all} \quad n > m$$

- is $\mathbf{2}n - \mathbf{1}$ an element of $O(n)$?
  - yes, let $M = \mathbf{2}$ and $m = \mathbf{0}$, then $\mathbf{2}n - \mathbf{1} < \mathbf{2}n$ for all $n > \mathbf{0}$

Informal Analysis of Merge Sort

- suppose the running time (the number of operations) of merge sort is a function of the number of elements to sort
  - let the function be $T(n)$
- merge sort works by splitting the list into two sub-lists (each about half the size of the original list) and sorting the sub-lists
  - this takes $2T(n/2)$ running time
- then the sub-lists are merged
  - this takes $O(n)$ running time
- total running time $T(n) = 2T(n/2) + O(n)$

Solving the Recurrence Relation

$T(n)$ approaches...

$$
\begin{aligned}
T(n) &\to 2T(n/2) + O(n) \\
&\approx 2T(n/2) + n \\
&= 2[\ 2T(n/4) + n/2\ ] + n \\
&= 4T(n/4) + 2n \\
&= 4[\ 2T(n/8) + n/4\ ] + 2n \\
&= 8T(n/8) + 3n \\
&= 8[\ 2T(n/16) + n/8\ ] + 3n \\
&= 16T(n/16) + 4n \\
&= 2^k T(n/2^k) + kn
\end{aligned}
$$

Solving the Recurrence Relation

$$T(n) \quad = \qquad \mathbf{2}^k T(\underline{n/\mathbf{2}^k}) + kn$$

- for a list of length **1** we know $T(\mathbf{1}) = \mathbf{1}$
  - if we can substitute $T(1)$ into the right-hand side of $T(n)$ we might be able to solve the recurrence

$$\underline{n/\mathbf{2}^k} = \mathbf{1} \Rightarrow \mathbf{2}^k = n \Rightarrow k = \log(n)$$

Solving the Recurrence Relation

$$
\begin{aligned}
T(n) &= \mathbf{2}^{\log(n)} T(n/\mathbf{2}^{\log(n)}) + n\log(n) \\
&= n\,T(\mathbf{1}) + n\log(n) \\
&= n + n\log(n) \\
&\in n\log(n)
\end{aligned}
$$

Is Merge Sort Efficient?

- consider a simpler (non-recursive) sorting algorithm called insertion sort

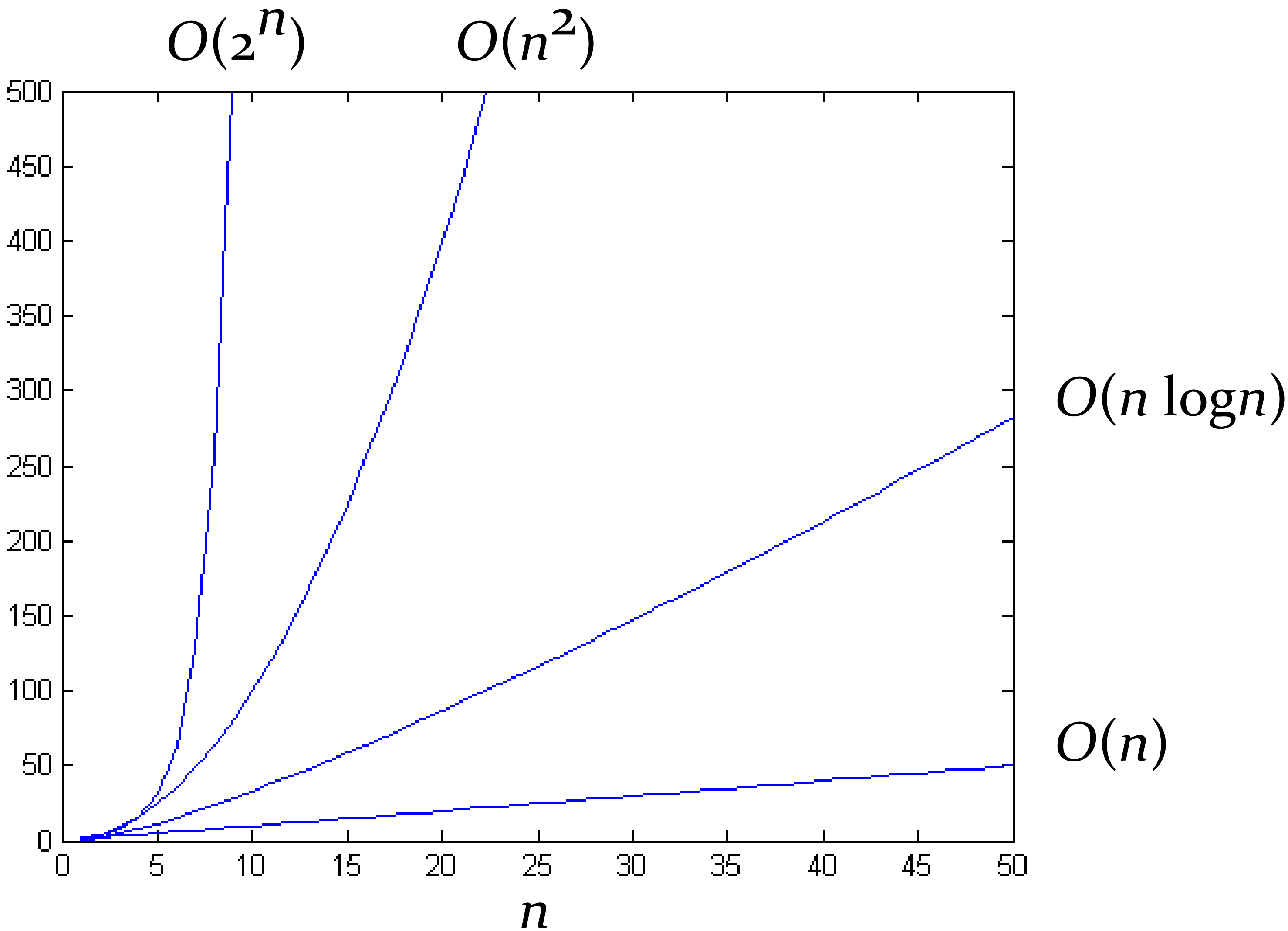
```
// to sort an array a[0]..a[n-1]                          not Java!
for i = 0 to (n-1) {
  k = index of smallest element in sub-array a[i]..a[n-1]
  swap a[i] and a[k]
}
```

```
for i = 0 to (n-1) {                                      not Java!
  for j = (i+1) to (n-1) {
    if (a[j] < a[i]) {                                    1 comparison +
      k = j;                                              1 assignment
    }
  }
  tmp = a[i];   a[i] = a[k];   a[k] = tmp;                3 assignments
}
```

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{n-1}\left(\left(\sum_{j=(i+1)}^{n-1} 2\right)+3\right) \\
&= \sum_{i=0}^{n-1}\left(2(n-i-1)\right) \quad + \quad 3n \\
&= 2\sum_{i=0}^{n-1} n \quad - \quad 2\sum_{i=0}^{n-1} i \quad - \quad 2\sum_{i=0}^{n-1} 1 \quad + \quad 3n \\
&= 2n^2 \quad - \quad 2\frac{n(n-1)}{2} \quad - \quad 2n \quad + \quad 3n \\
&= 2n^2 \quad - \quad n^2 \quad + \quad n \quad - \quad 2n \quad + \quad 3n
\end{aligned}
$$

$$
= n^2 \quad + \quad 2n \quad \in O\left(n^2\right)
$$

Comparing Rates of Growth



Comments

- big O complexity tells you something about the running time of an algorithm as the size of the input, n, approaches infinity
  - we say that it describes the limiting, or asymptotic, running time of an algorithm
- for small values of n it is often the case that a less efficient algorithm (in terms of big O) will run faster than a more efficient one
  - insertion sort is typically faster than merge sort for short lists of numbers

Revisiting the Fibonacci Numbers

- the recursive implementation based on the definition of the Fibonacci numbers is inefficient

```
public static int fibonacci(int n) {
  if (n == 0) {
   return 0;
  }
  else if (n == 1) {
   return 1;
  }
  int f = fibonacci(n - 1) + fibonacci(n - 2);
  return f;
}
```

- how inefficient is it?
- let $T(n)$ be the running time to compute the nth Fibonacci number
  - $T(0) = T(1) = 1$
  - $T(n)$ is a recurrence relation

$$
\begin{aligned}
T(n) \quad & \to T(n-1)+T(n-2) \\
& = \big(T(n-2)+T(n-3)\big)+T(n-2) \\
& = 2T(n-2)+T(n-3) \\
& > 2T(n-2) \\
& > 2\big(2T(n-4)\big) = 4T(n-4) \\
& > 4\big(2T(n-6)\big) = 8T(n-6) \\
& > 8\big(2T(n-8)\big) = 16T(n-8) \\
& > 2^{k}T(n-2k)
\end{aligned}
$$

Solving the Recurrence Relation

$$T(n) \quad > \qquad 2^k T(\underline{n - 2k})$$

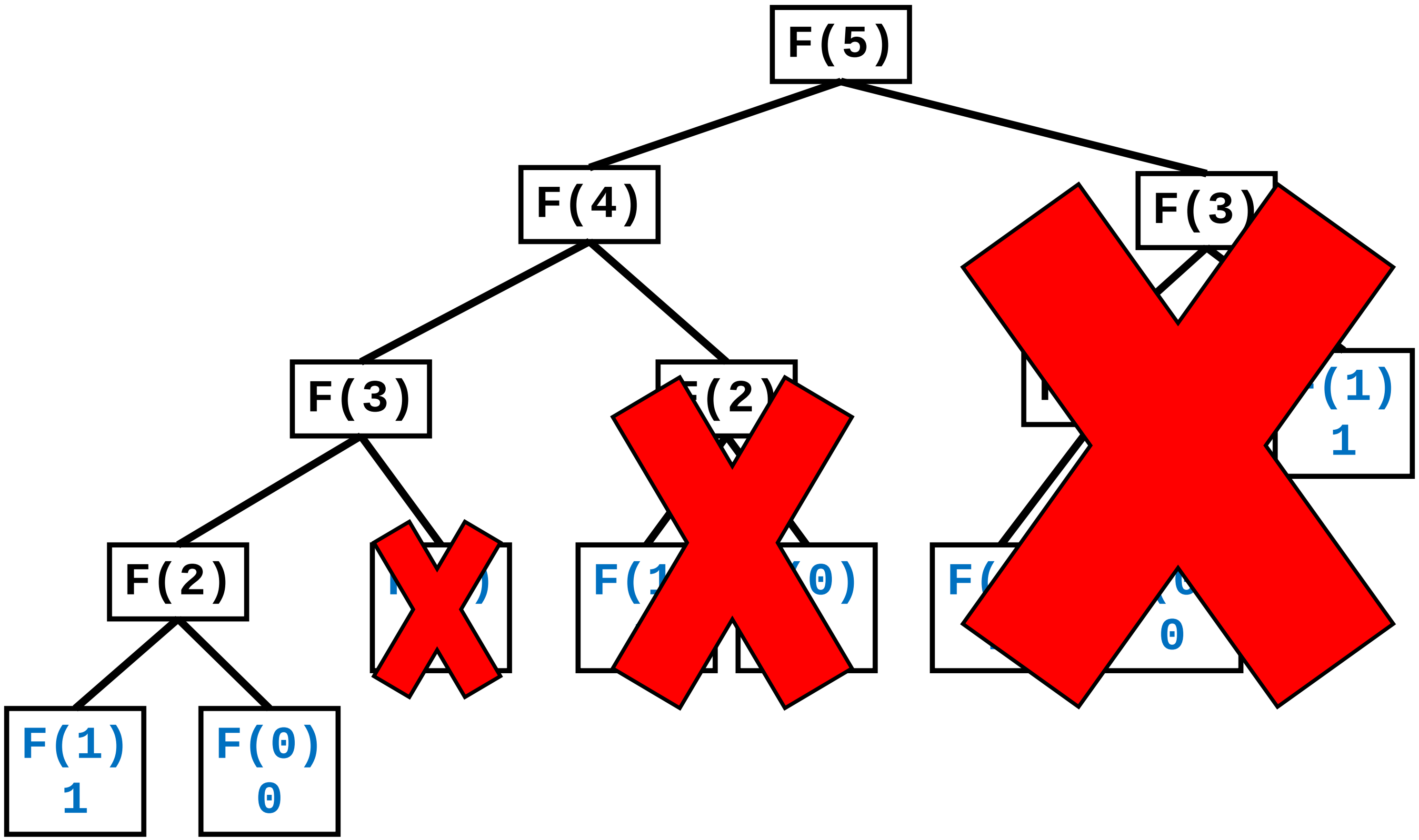
- we know $T(1) = 1$
  - if we can substitute $T(1)$ into the right-hand side of $T(n)$ we might be able to solve the recurrence

$$\underline{n - 2k} = 1 \;\Rightarrow\; 1 + 2k = n \Rightarrow k = (\mathrm{n} - 1)/2$$

$$T(n) > 2^k T(n-2k) \quad = \quad 2^{(n-1)/2} T(1) \quad = \quad 2^{(n-1)/2} \quad \in \quad O(2^n)$$

An Efficient Fibonacci Algorithm

- an $O(n)$ algorithm exists that computes all of the Fibonacci numbers from $f(0)$ to $f(n)$



- create an array of length $(n + 1)$ and sequentially fill in the array values
  - $O(n)$

```
// pre. n >= 0
public static int[] fibonacci(int n) {
  int[] f = new int[n + 1];
  f[0] = 0;
  f[1] = 1;
  for (int i = 2; i < n + 1; i++) {
    f[i] = f[i - 1] + f[i - 2];
  }
  return f;
}
```

Closing Question

- the recursive Fibonacci and merge sort algorithms can be illustrated using a call tree
  - merge sort is actually 2 trees; one to split and one to merge

- why is the Fibonacci algorithm $O(2^n)$ and merge sort $O(n \log n)$?