Numerical integration

Numerical integration

- numerical integration attempts to estimate the value of a definite integral without solving for the indefinite integral; i.e.,
  - estimate the value of $I = \int_a^b f(x)dx$

    without solving for $F(x) = \int f(x)dx$

- recall the first fundamental theorem of calculus

$$I = \int_a^b f(x)dx = F(b) - F(a)$$

Numerical integration

- why would you want to do this?
  - many indefinite integrals cannot be written in terms of elementary functions; e.g., the error function

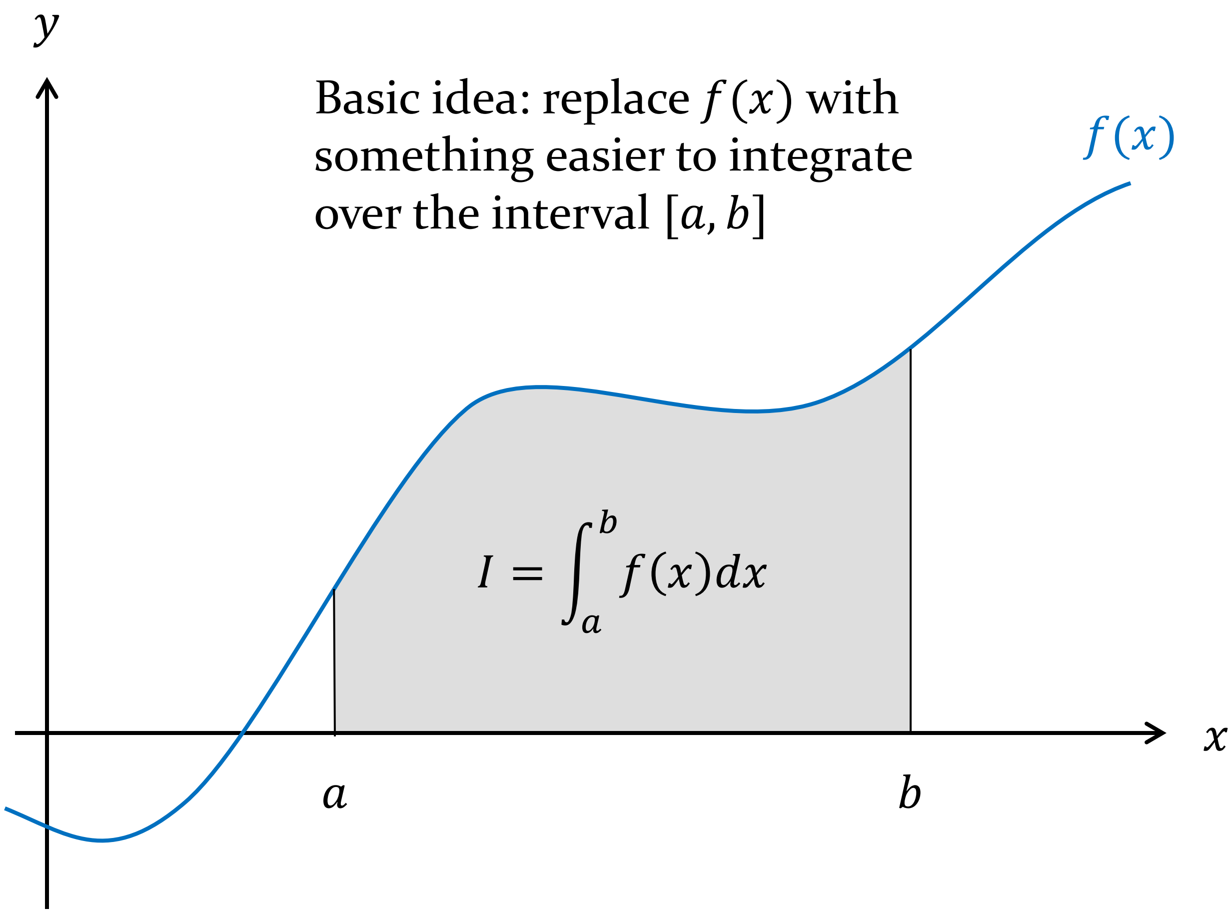
$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

  - the function $f(x)$ is not known; e.g., you only have measurements of some unknown function $f(x)$
    - inertial measurement units (IMU) measure accelerations, the acceleration measurements are integrated to obtain velocity estimates, and the velocities are integrated to obtain position
  - the indefinite integral is known, but difficult or computationally expensive to evaluate

Aside

- many symbolic math programs can solve for the indefinite integral if the integral can be expressed in terms of elementary functions
- how do they do this?
  - http://en.wikipedia.org/wiki/Risch_algorithm

Numerical integration



Rectangle (or midpoint) rule

- replaces $f(x)$ with a constant over the interval $[a, b]$
  - this approximates the area under $f(x)$ with a rectangle
  - height of the rectangle

$$f(\frac{a+b}{2})$$

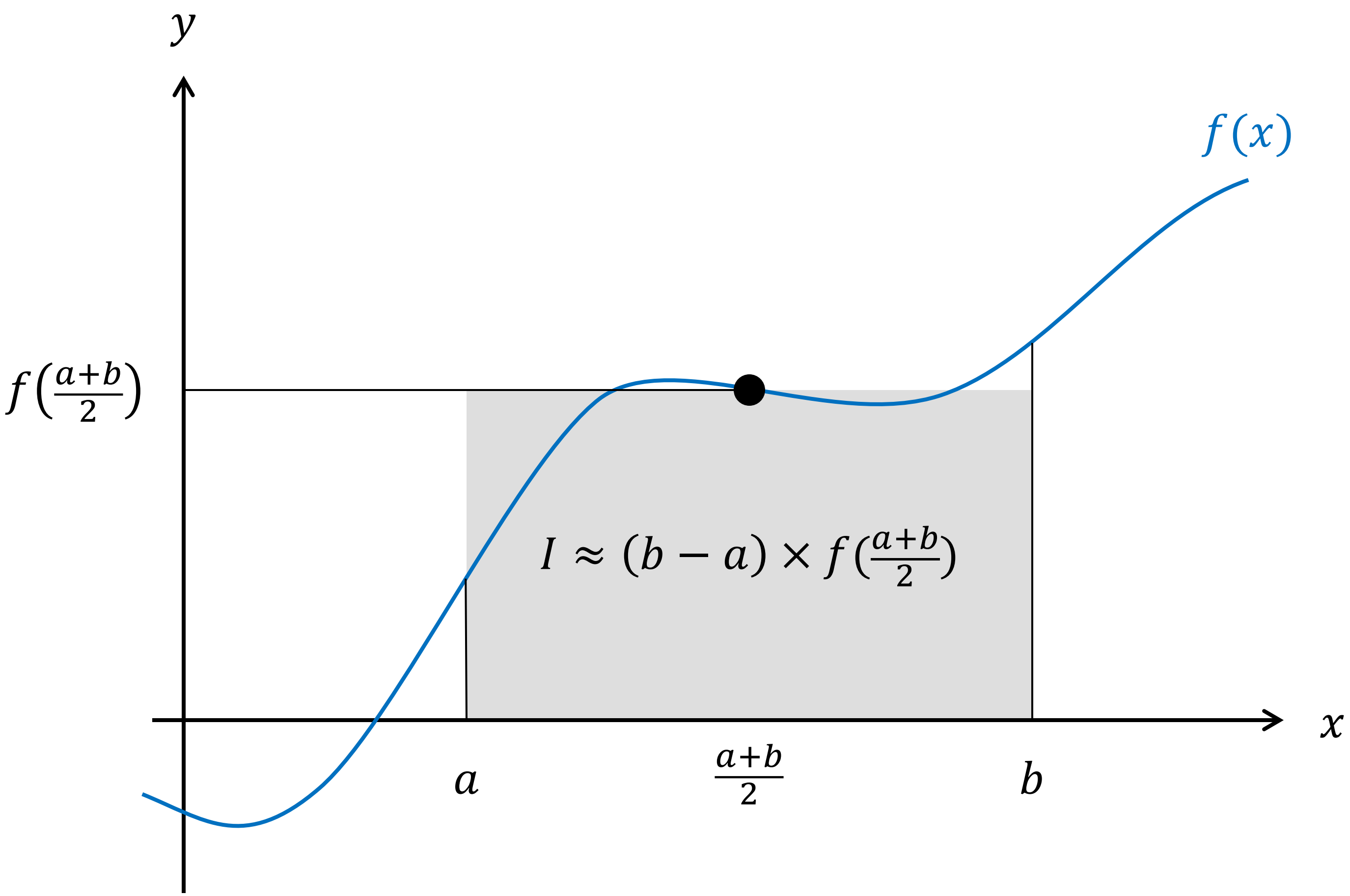
  - width of the rectangle

$$b - a$$

  - area of the rectangle

$$(b - a) \times f(\frac{a+b}{2})$$

Rectangle (or midpoint) rule



Trapezoid rule

- replaces $f(x)$ with a line over the interval $[a, b]$
  - this approximates the area under $f(x)$ with a trapezoid
  - sides of the trapezoid

$$f(a) \quad \text{and} \quad f(b)$$

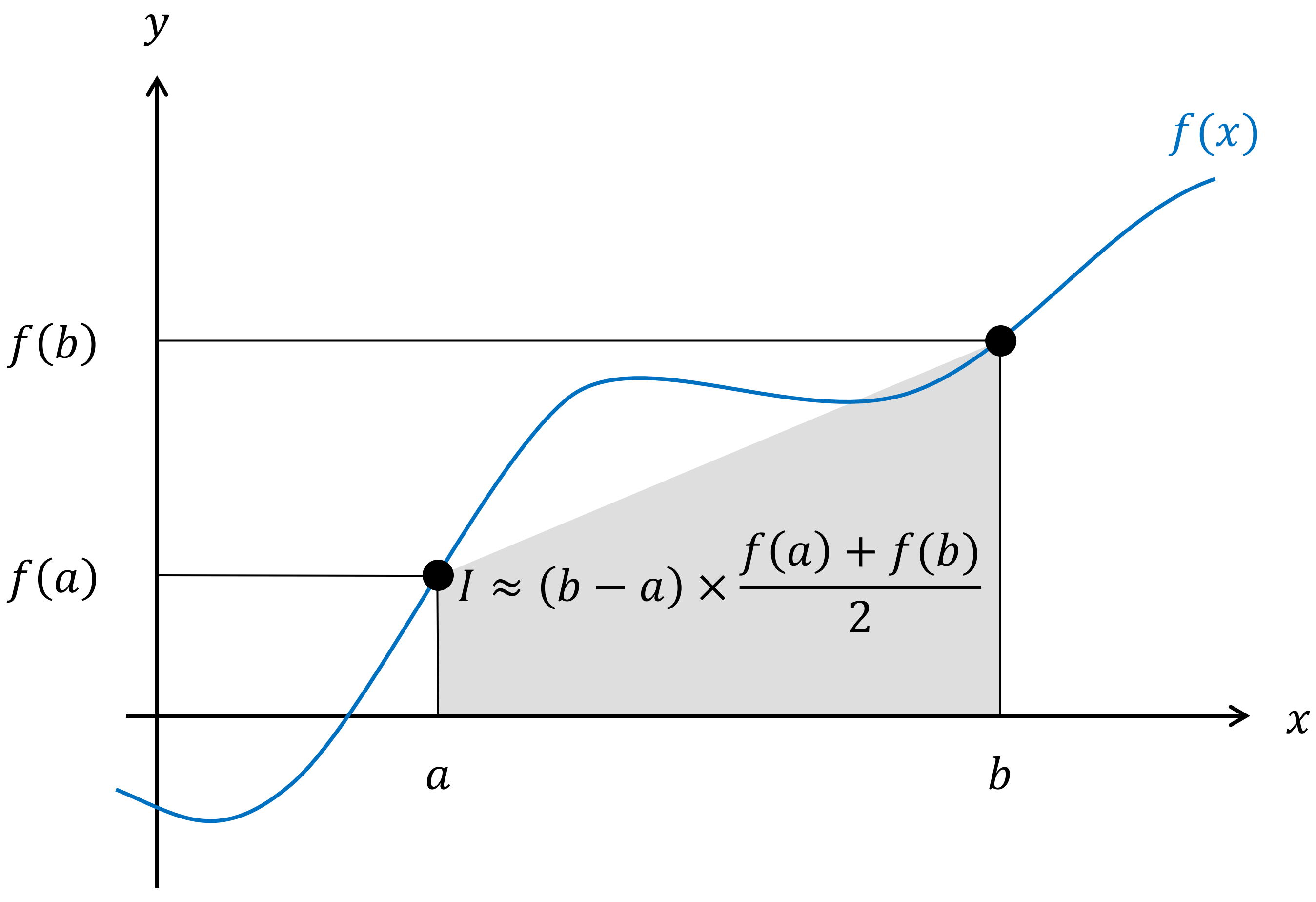
  - width of the trapezoid

$$b - a$$

  - area of the trapezoid

$$(b - a) \times \frac{f(a) + f(b)}{2}$$

Trapezoid rule



Simpson's rule

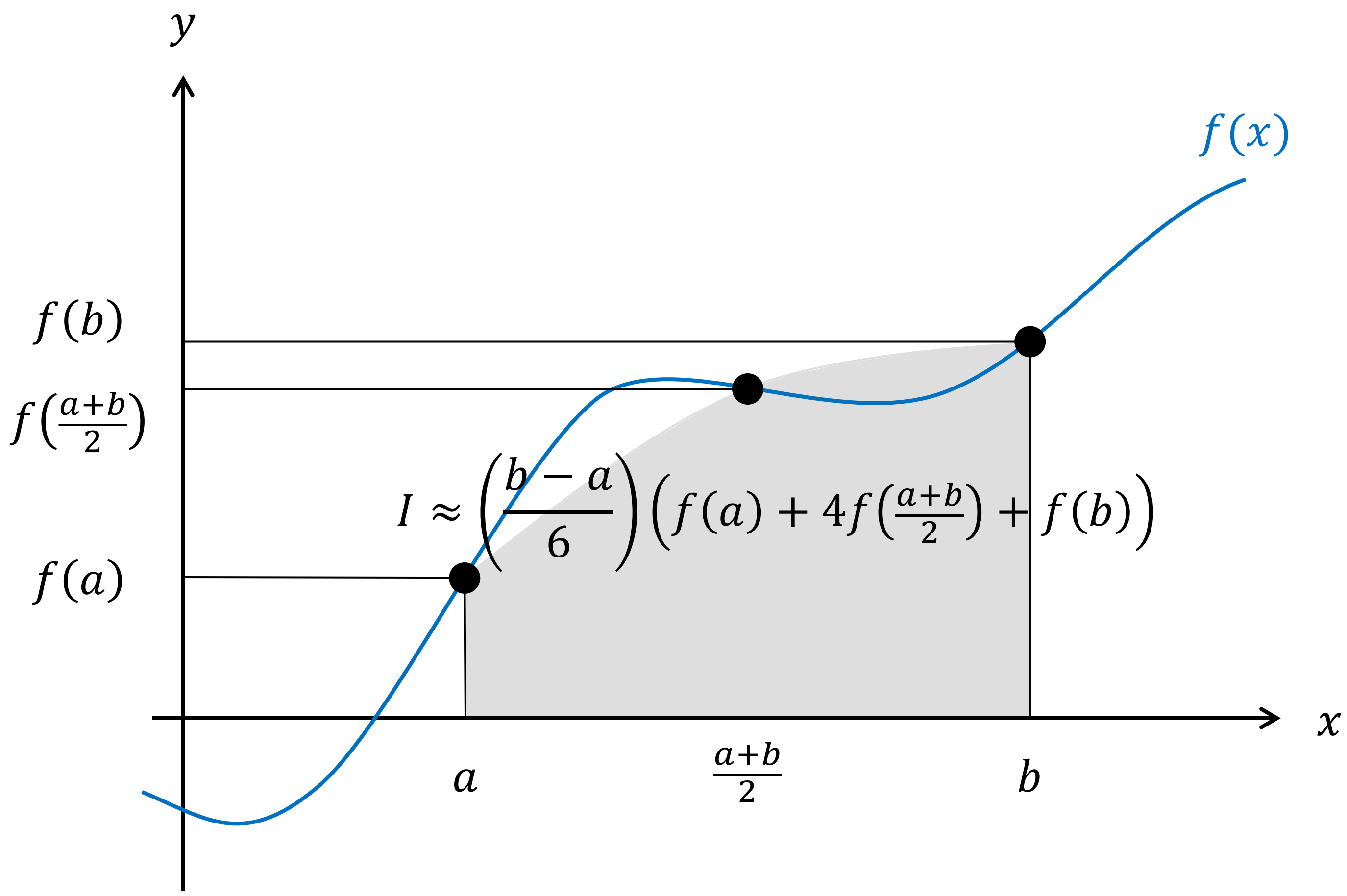
- replaces $f(x)$ with a quadratic over the interval $[a, b]$
  - this approximates the area under $f(x)$ as the area under a parabola
  - parabola passes through the points

$$f(a) \quad \text{and} \quad f\left(\frac{a+b}{2}\right) \quad \text{and} \quad f(b)$$

  - area under the parabola

$$\left(\frac{b-a}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

Simpson's rule

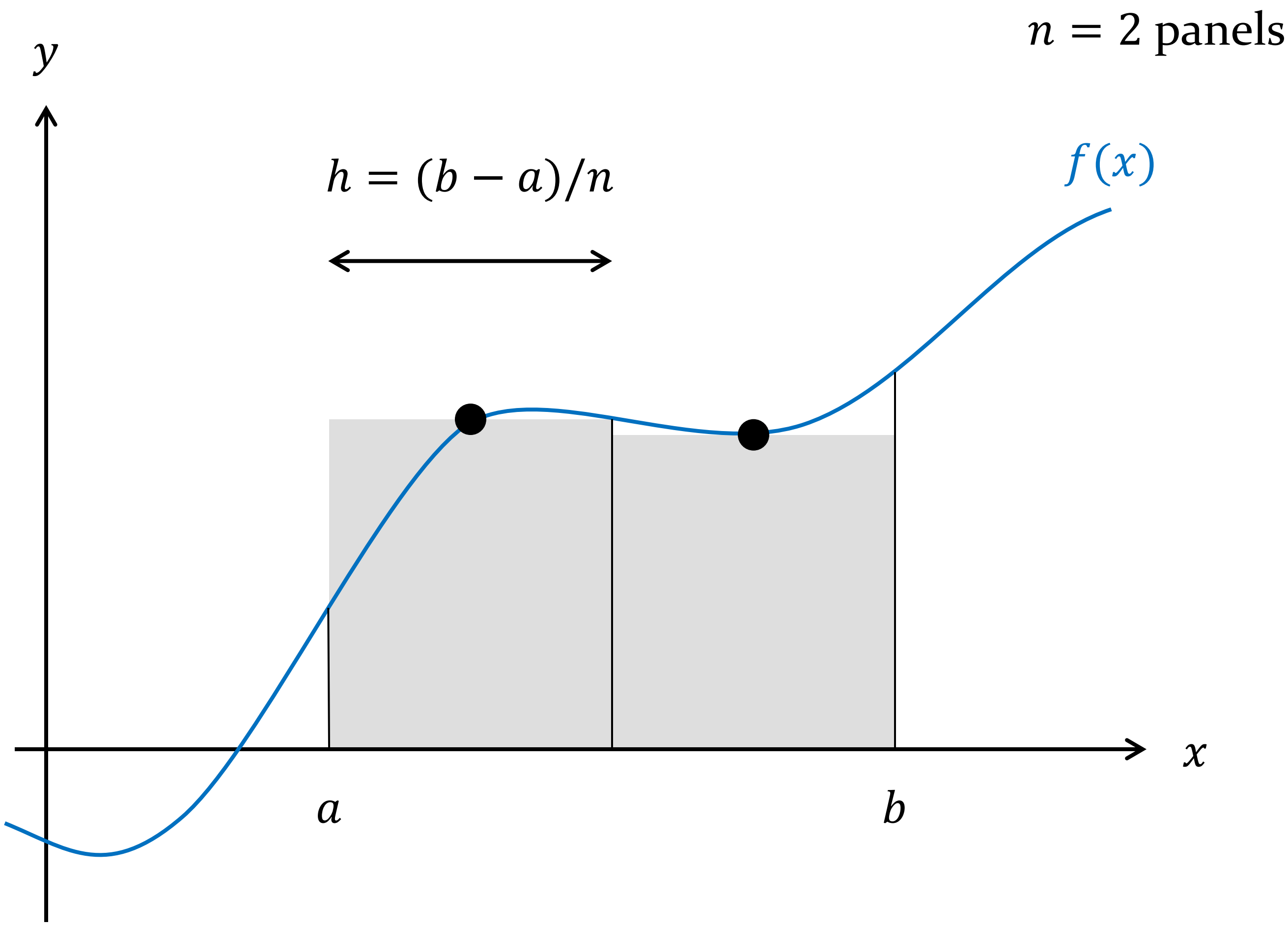


Composite rules

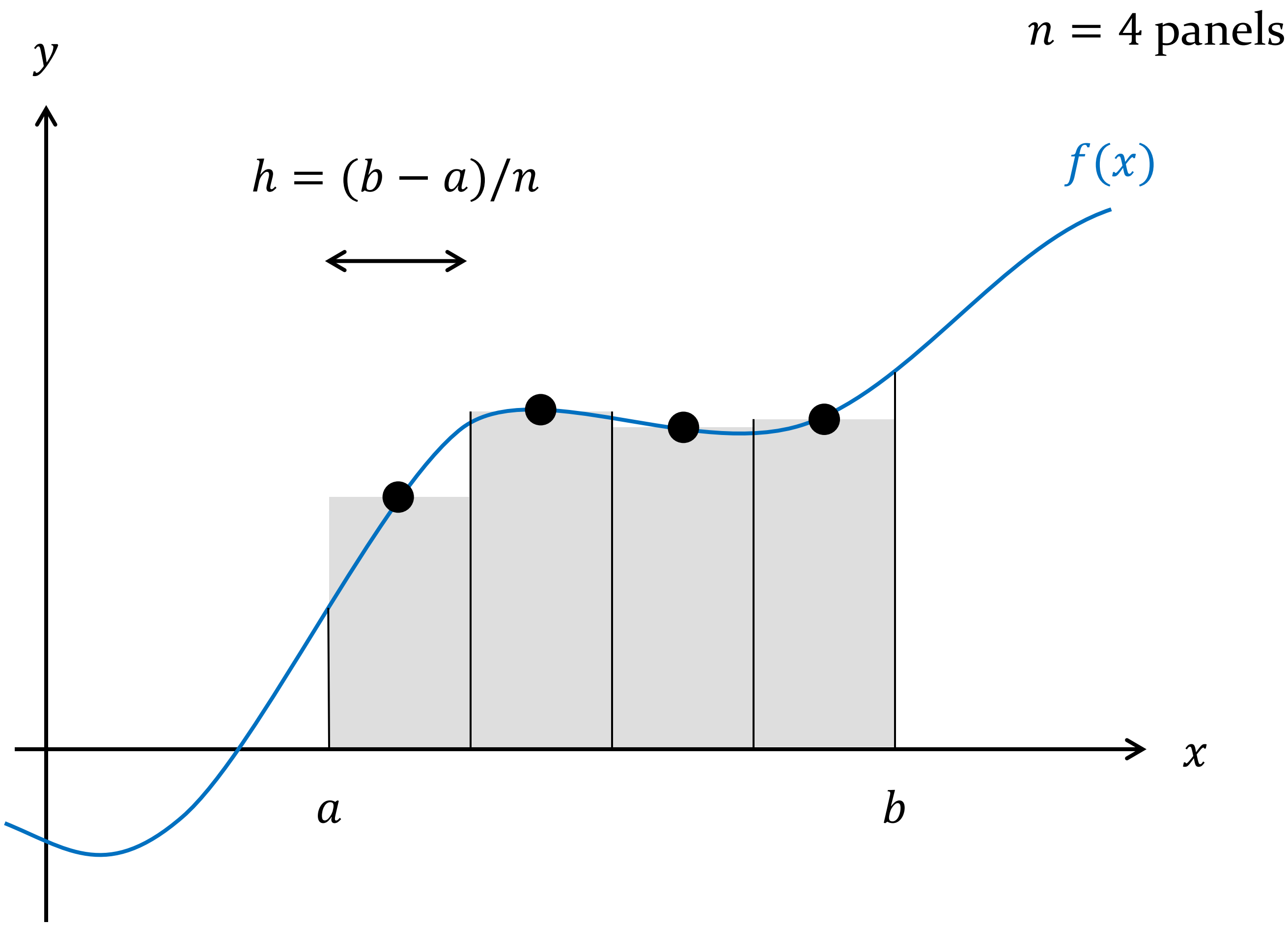
- the midpoint, trapezoid, and Simpson's rules are called *basic* rules
- the composite rule subdivides the range $[a, b]$ using n points to form $(n - 1)$ panels
  - a basic rule is then applied to each panel

- the value of the integral is the sum of the area of the panels

Composite rectangle (or midpoint) rule

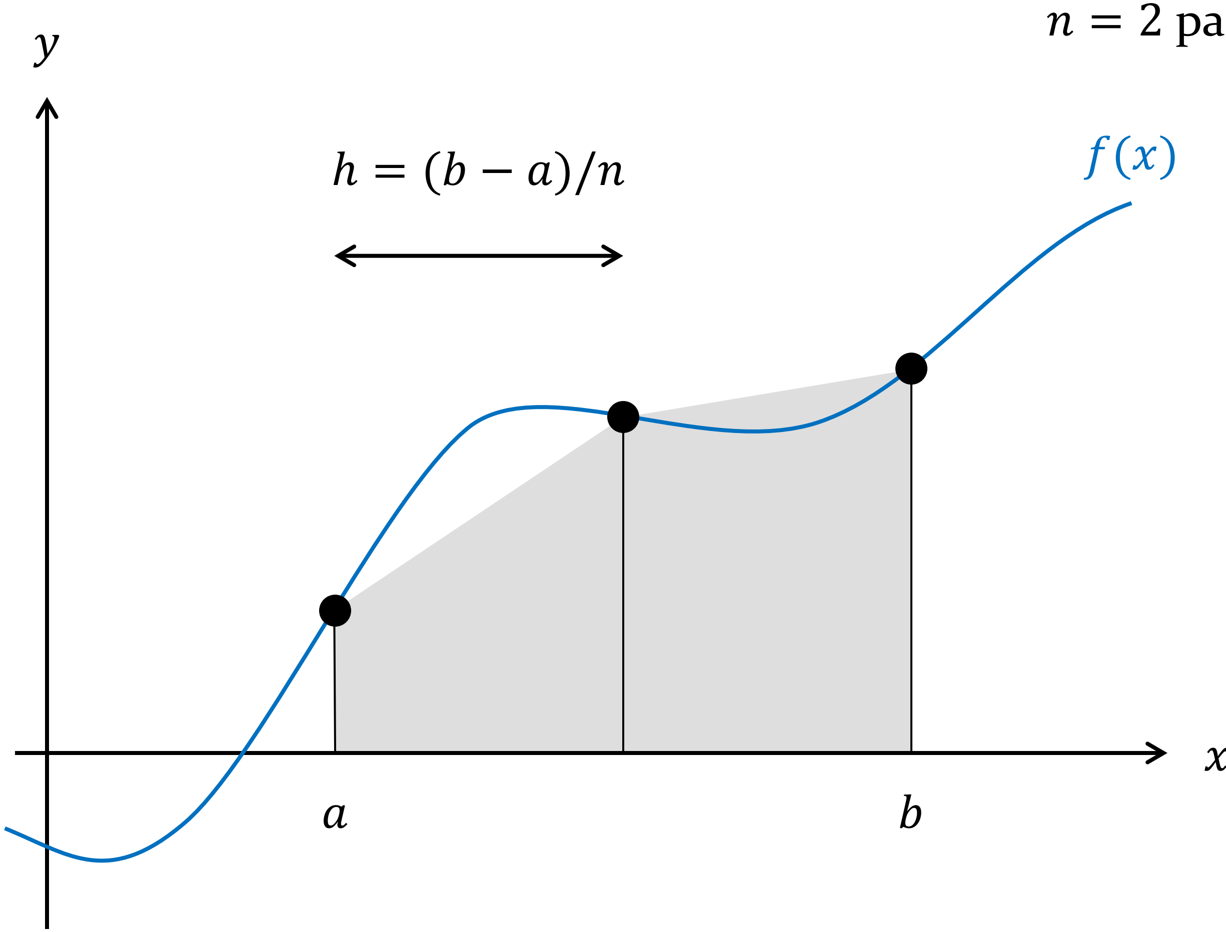


Composite rectangle (or midpoint) rule

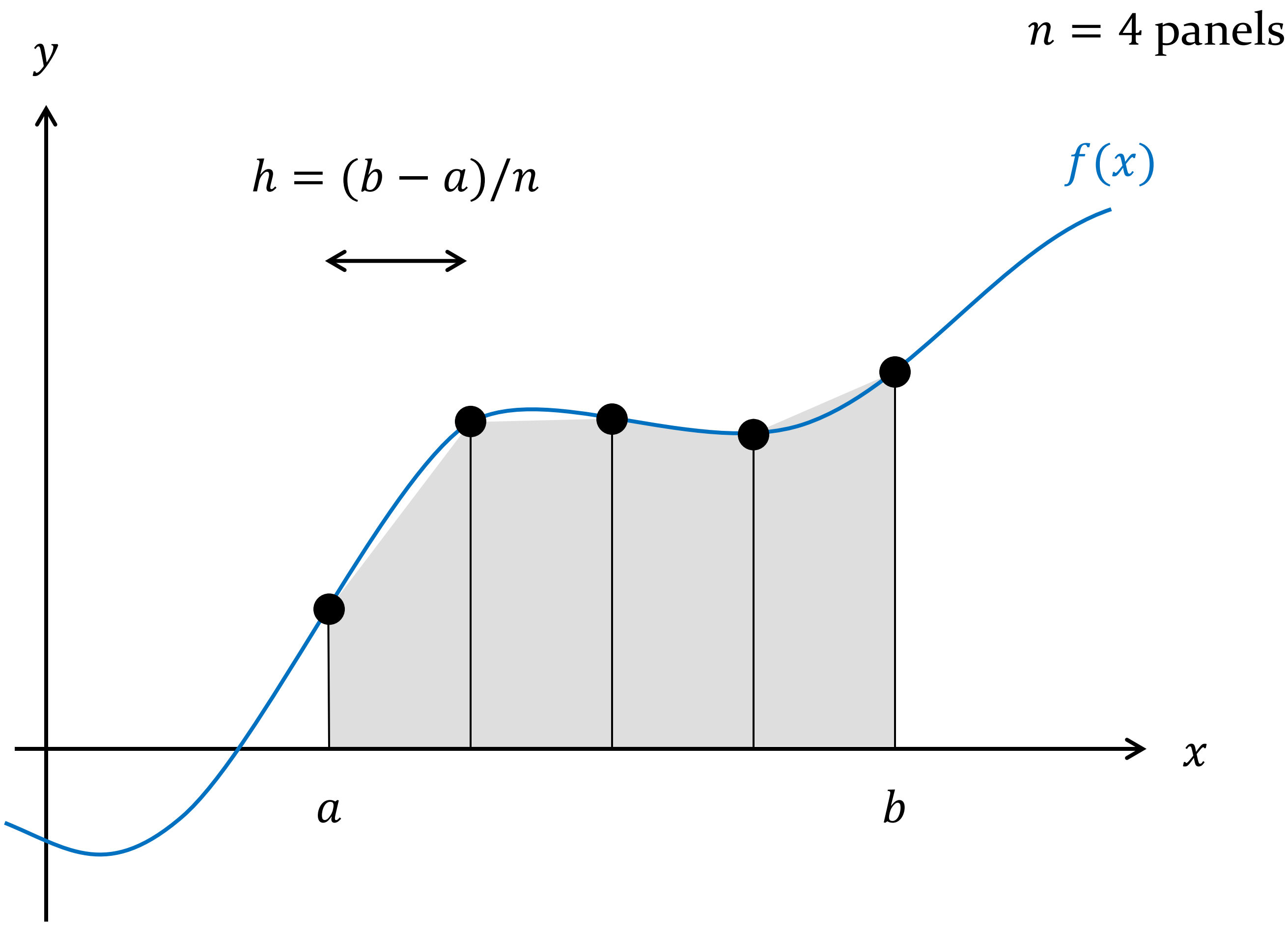


Composite trapezoid rule

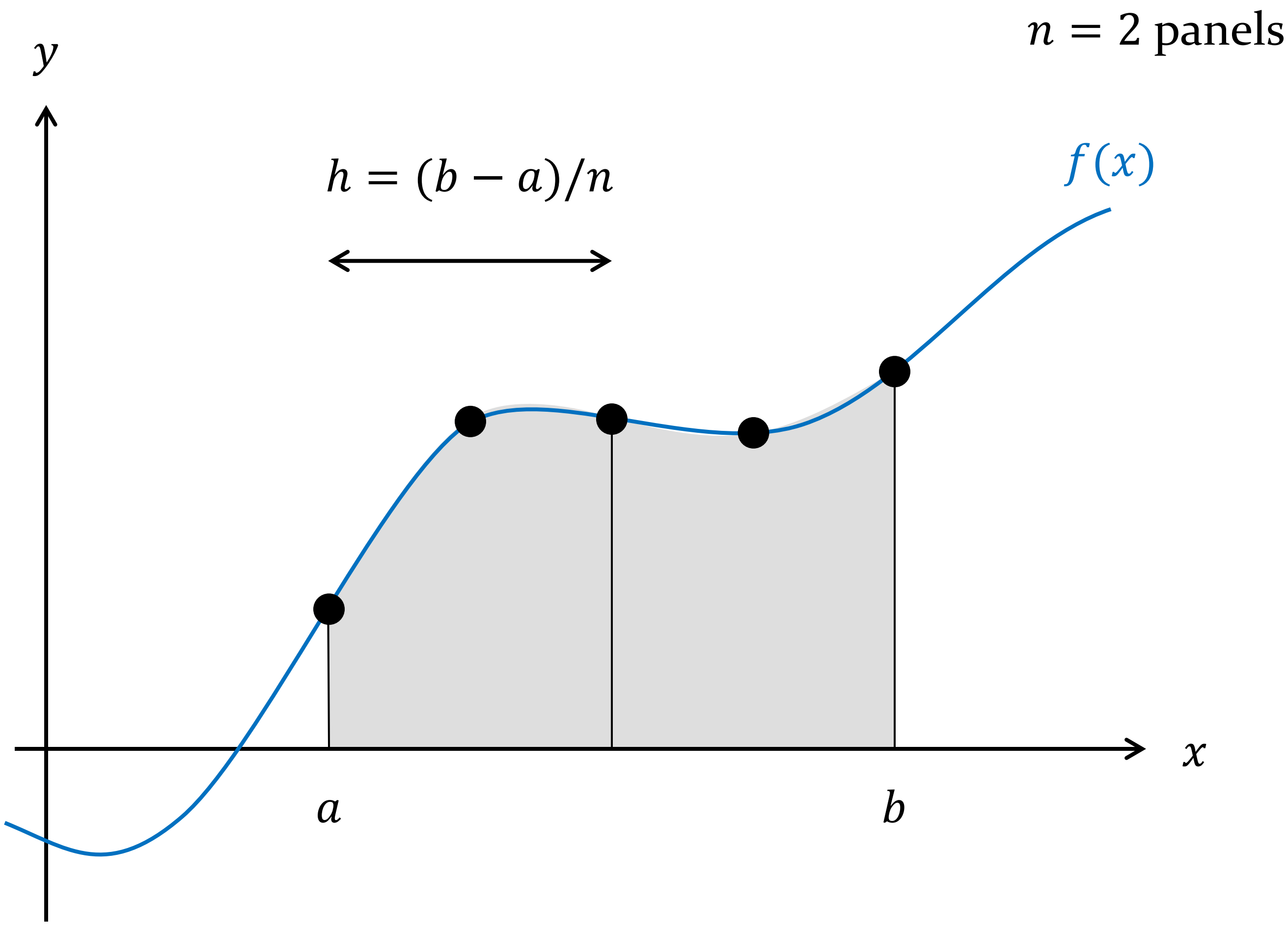
$n = 2$ panels



Composite trapezoid rule



Composite Simpson's rule



In class exercise

- implement composite rectangle, trapezoid, and Simpson's rule
- test implementations using known integrals

Numerical integration in MATLAB

- MATLAB provides functions for integration using
  - the trapezoidal rule
    - **`trapz`**
  - a more sophisticated composite rule (global adaptive quadrature)
    - **`integral`**, **`integral2`**, **`integral3`**