Recursion

notes Chapter 8

Printing n of Something

- suppose you want to implement a method that prints out n copies of a string

```
public static void printIt(String s, int n) {
  for(int i = 0; i < n; i++) {
    System.out.print(s);
  }
}
```

A Different Solution

- alternatively we can use the following algorithm:
  1. if n == 0 done, otherwise
     I. print the string once
     II. print the string (n – 1) more times

```
public static void printItToo(String s, int n) {
  if (n == 0) {
    return;
  }
  else {
    System.out.print(s);
    printItToo(s, n - 1);     // method invokes itself
  }
}
```

Recursion

- a method that calls itself is called a *recursive* method
- a recursive method solves a problem by repeatedly reducing the problem so that a base case can be reached

```
printItToo("*", 5)
*printItToo ("*", 4)
**printItToo ("*", 3)
***printItToo ("*", 2)
****printItToo ("*", 1)
*****printItToo ("*", 0) base case
*****
```

Notice that the number of times the string is printed decreases after each recursive call to printIt

Notice that the base case is eventually reached.

Infinite Recursion

- if the base case(s) is missing, or never reached, a recursive method will run forever (or until the computer runs out of resources)

```
public static void printItForever(String s, int n) {
  // missing base case; infinite recursion
  System.out.print(s);
  printItForever(s, n - 1);
}
```

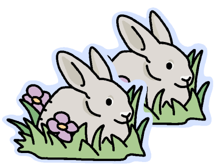
```
printItForever("*", 1)
* printItForever("*", 0)
** printItForever("*", -1)
*** printItForever("*", -2) ...........
```

Climbing a Flight of n Stairs

- not Java

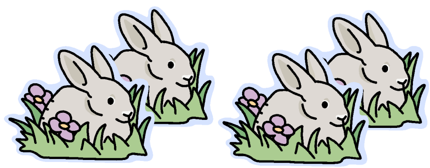
```
climb(n) :
if n == 0
  done
else
  step up 1 stair
  climb(n – 1);
end
```

Rabbits



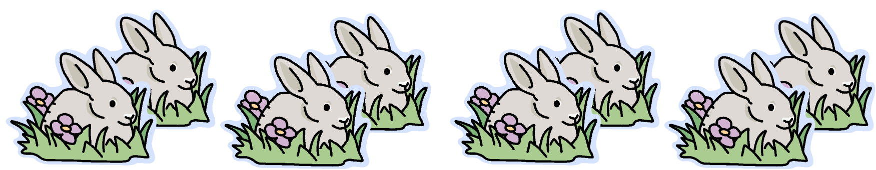
Month 0: 1 pair

0 additional pairs



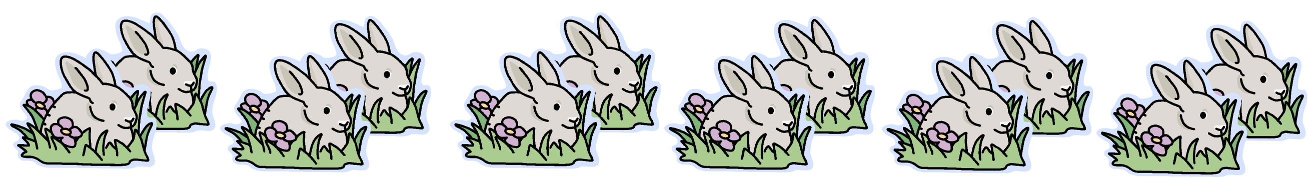
Month 1: first pair
makes another pair

1 additional pair



Month 2: each pair
makes another pair;
oldest pair dies

1 additional pair



2 additional pairs

Month 3: each pair
makes another pair;
oldest pair dies

Fibonacci Numbers

- the sequence of additional pairs
  - `0, 1, 1, 2, 3, 5, 8, 13, ...`

  are called Fibonacci numbers

- base cases
  - **`F(0) = 0`**
  - **`F(1) = 1`**
- recursive definition
  - **`F(n) = F(n – 1) +  F(n – 2)`**

Recursive Methods & Return Values

- a recursive method can return a value
- example: compute the nth Fibonacci number

```
public static int fibonacci(int n) {
  if (n == 0) {
    return 0;
  }
  else if (n == 1) {
    return 1;
  }
  else {
   int f = fibonacci(n - 1) + fibonacci(n - 2);
   return f;
  }
}
```

Recursive Methods & Return Values

- example: write a recursive method **`countZeros`** that counts the number of zeros in an integer number **`n`**
  - **`10305060700002L`** has 8 zeros

- trick: examine the following sequence of numbers
  1. **`10305060700002`**
  2. **`10305060700000`**
  3. **`1030506070000`**
  4. **`103050607000`**
  5. **`103050607`**
  6. **`1030506 ...`**

Recursive Methods & Return Values

- not Java:

```
countZeros(n) :
if the last digit in n is a zero
  return 1 + countZeros(n / 10)
else
  return countZeros(n / 10)
```

- don't forget to establish the base case(s)
  - when should the recursion stop? when you reach a single digit (not zero digits; you never reach zero digits!)
    - base case #1 : `n == 0`
      - `return 1`
    - base case #2 : `n != 0 && n < 10`
      - `return 0`

```java
public static int countZeros(long n) {

  if(n == 0L) {  // base case 1
    return 1;
  }
  else if(n < 10L) {  // base case 2
    return 0;
  }

  boolean lastDigitIsZero = (n % 10L == 0);
  final long m = n / 10L;
  if(lastDigitIsZero) {
    return 1 + countZeros(m);
  }
  else {
    return countZeros(m);
  }
}
```

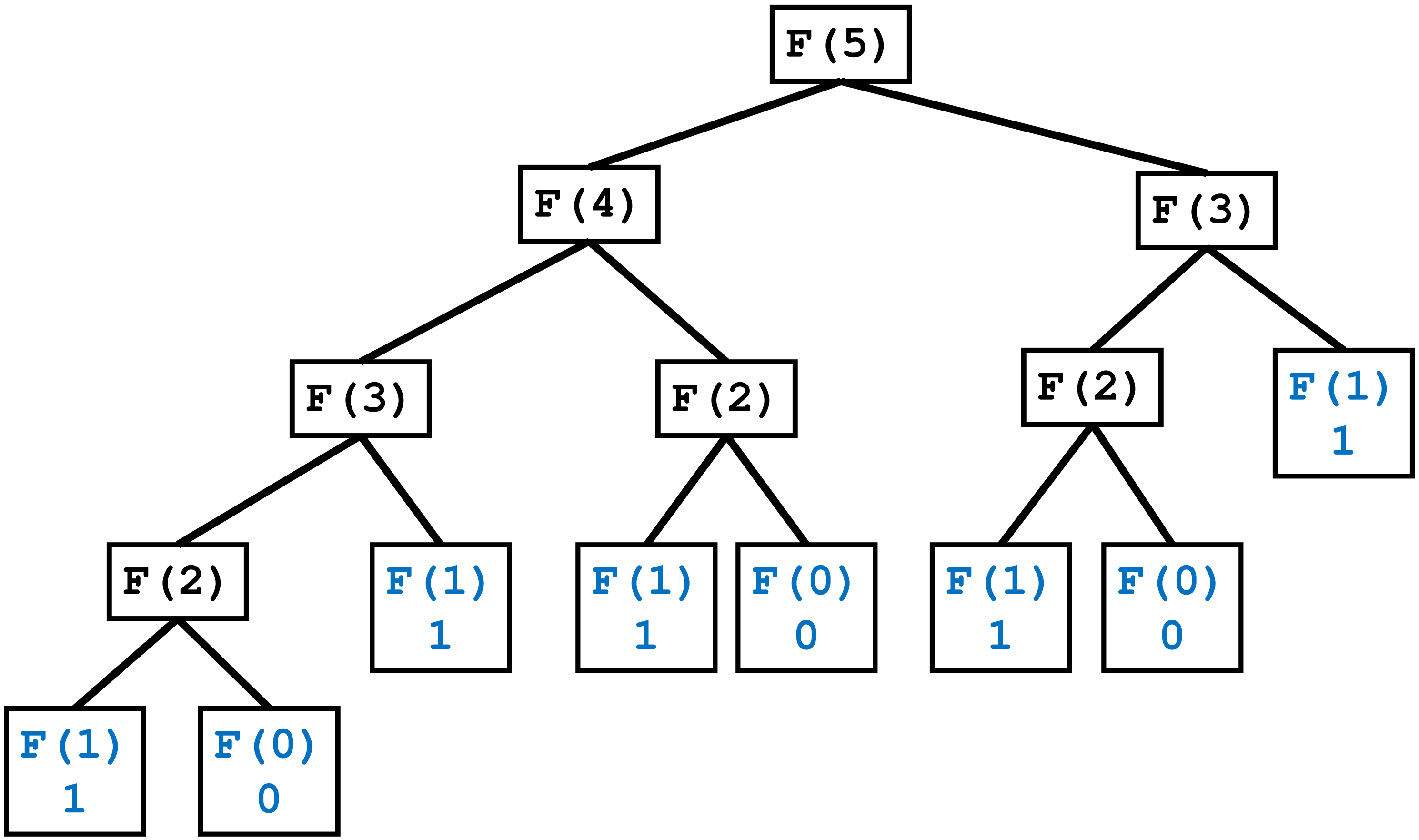
countZeros Call Stack

`callZeros( 800410L )`

last in first out

`callZeros( 8L )`	`0`
`callZeros( 80L )`	`1 + 0`
`callZeros( 800L )`	`1 + 1 + 0`
`callZeros( 8004L )`	`0 + 1 + 1 + 0`
`callZeros( 80041L )`	`0 + 0 + 1 + 1 + 0`
`callZeros( 800410L )`	`1 + 0 + 0 + 1 + 1 + 0`
	`= 3`

Fibonacci Call Tree



Compute Powers of 10

- write a recursive method that computes 10^n for any integer value **`n`**
- recall:
  - $10^0 = 1$
  - $10^n = 10 * 10^{n-1}$
  - $10^{-n} = 1 / 10^n$

```java
public static double powerOf10(int n) {
  if (n == 0) {
    // base case
    return 1.0;
  }
  else if (n > 0) {
    // recursive call for positive n
    return 10.0 * powerOf10(n - 1);
  }
  else {
    // recursive call for negative n
    return 1.0 / powerOf10(-n);
  }
}
```

Proving Correctness and Termination

- to show that a recursive method accomplishes its goal you must prove:
  1. that the base case(s) and the recursive calls are correct
  2. that the method terminates

Proving Correctness

- to prove correctness:
  1. prove that each base case is correct
  2. assume that the recursive invocation is correct and then prove that each recursive case is correct

printItToo

```
public static void printItToo(String s, int n) {
  if (n == 0) {
    return;
  }
  else {
    System.out.print(s);
    printItToo(s, n - 1);
  }
}
```

Correctness of printItToo

1. (prove the base case) If `n == 0` nothing is printed; thus the base case is correct.
2. Assume that `printItToo(s, n-1)` prints the string `s` exactly `(n - 1)` times. Then the recursive case prints the string `s` exactly `(n - 1)+1 = n` times; thus the recursive case is correct.

Proving Termination

- to prove that a recursive method terminates:
  1. define the size of a method invocation; the size must be a non-negative integer number
  2. prove that each recursive invocation has a smaller size than the original invocation

Termination of printIt

1. **`printIt(s, n)`** prints **`n`** copies of the string **`s`**; define the size of **`printIt(s, n)`** to be **`n`**
2. The size of the recursive invocation **`printIt(s, n-1)`** is **`n-1`** (by definition) which is smaller than the original size **`n`**.

countZeros

```
public static int countZeros(long n) {

  if(n == 0L) {  // base case 1
    return 1;
  }
  else if(n < 10L) {  // base case 2
    return 0;
  }

  boolean lastDigitIsZero = (n % 10L == 0);
  final long m = n / 10L;
  if(lastDigitIsZero) {
    return 1 + countZeros(m);
  }
  else {
    return countZeros(m);
  }
}
```

Correctness of countZeros

1. (base cases) If the number has only one digit then the method returns **`1`** if the digit is zero and **`0`** if the digit is not zero; therefore, the base case is correct.
2. (recursive cases) Assume that **`countZeros(n/10L)`** is correct (it returns the number of zeros in the first **`(d - 1)`** digits of **`n`**). If the last digit in the number is zero, then the recursive case returns **`1 +`** the number of zeros in the first **`(d - 1)`** digits of **`n`**, otherwise it returns the number of zeros in the first **`(d - 1)`** digits of **`n`**; therefore, the recursive cases are correct.

Termination of countZeros

1. Let the size of **`countZeros(n)`** be **`d`** the number of digits in the number **`n`**.
2. The size of the recursive invocation **`countZeros(n/10L)`** is **`d-1`**, which is smaller than the size of the original invocation.

Decrease and Conquer

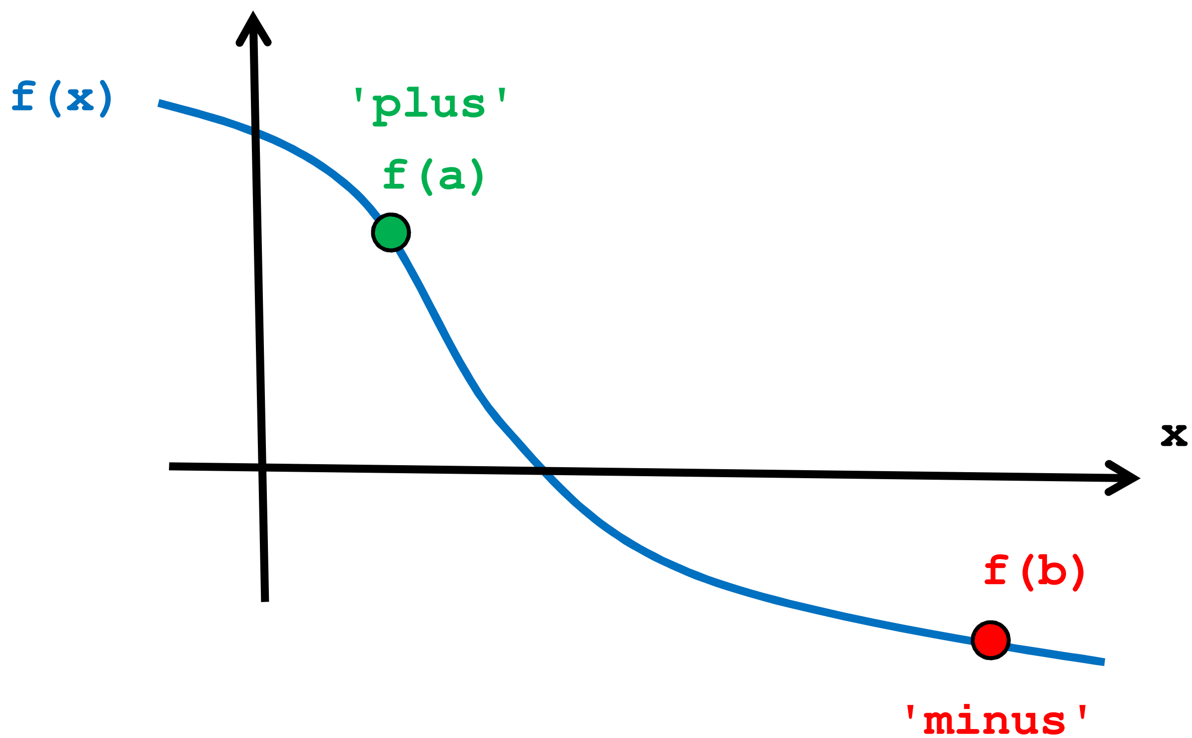
- a common strategy for solving computational problems
  - solves a problem by taking the original problem and converting it to *one* smaller version of the same problem
    - note the similarity to recursion
- decrease and conquer, and the closely related divide and conquer method, are widely used in computer science
  - allow you to solve certain complex problems easily
  - help to discover efficient algorithms

Root Finding

- suppose you have a mathematical function `f(x)` and you want to find x_0 such that $f(x_0) = 0$
  - why would you want to do this?
  - many problems in computer science, science, and engineering reduce to optimization problems
    - find the shape of an automobile that minimizes aerodynamic drag
    - find an image that is similar to another image (minimize the difference between the images)
    - find the sales price of an item that maximizes profit
  - if you can write the optimization criteria as a function `g(x)` then its derivative `f(x) = dg/dx = 0` at the minimum or maximum of `g` (as long as `g` has certain properties)

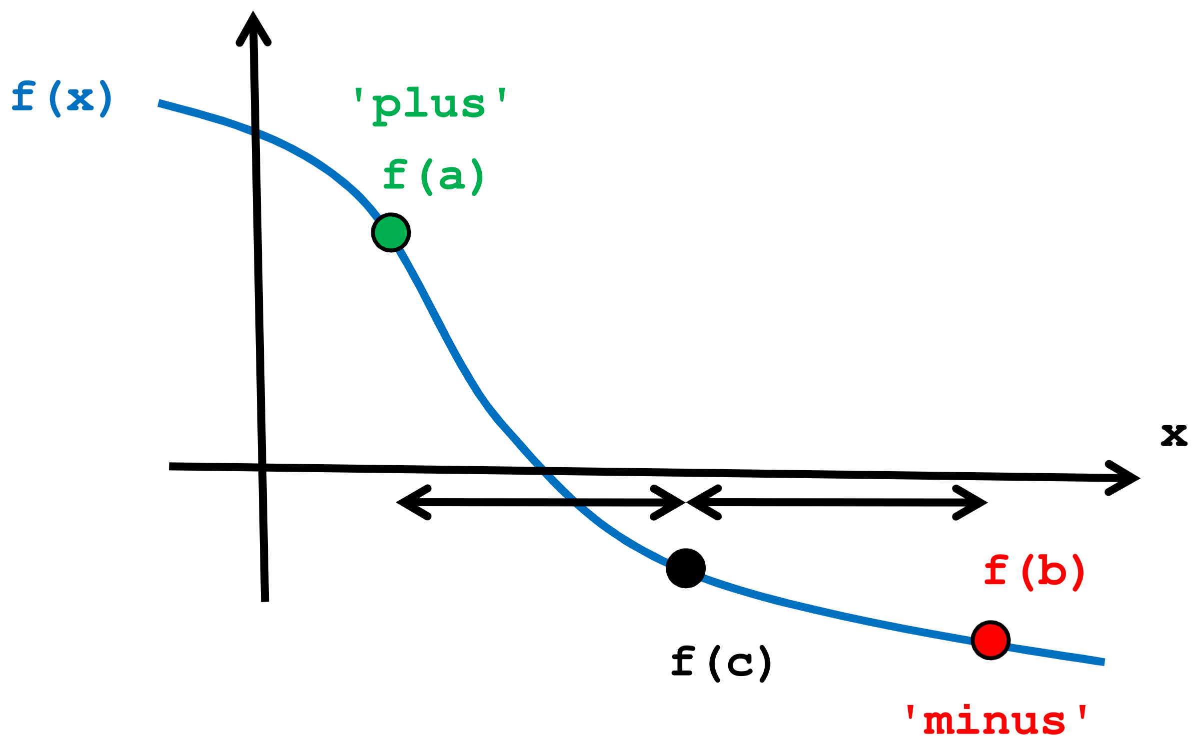
Bisection Method

- suppose you can evaluate **`f(x)`** at two points **`x = a`** and **`x = b`** such that
  - **`f(a) > 0`**
  - **`f(b) < 0`**



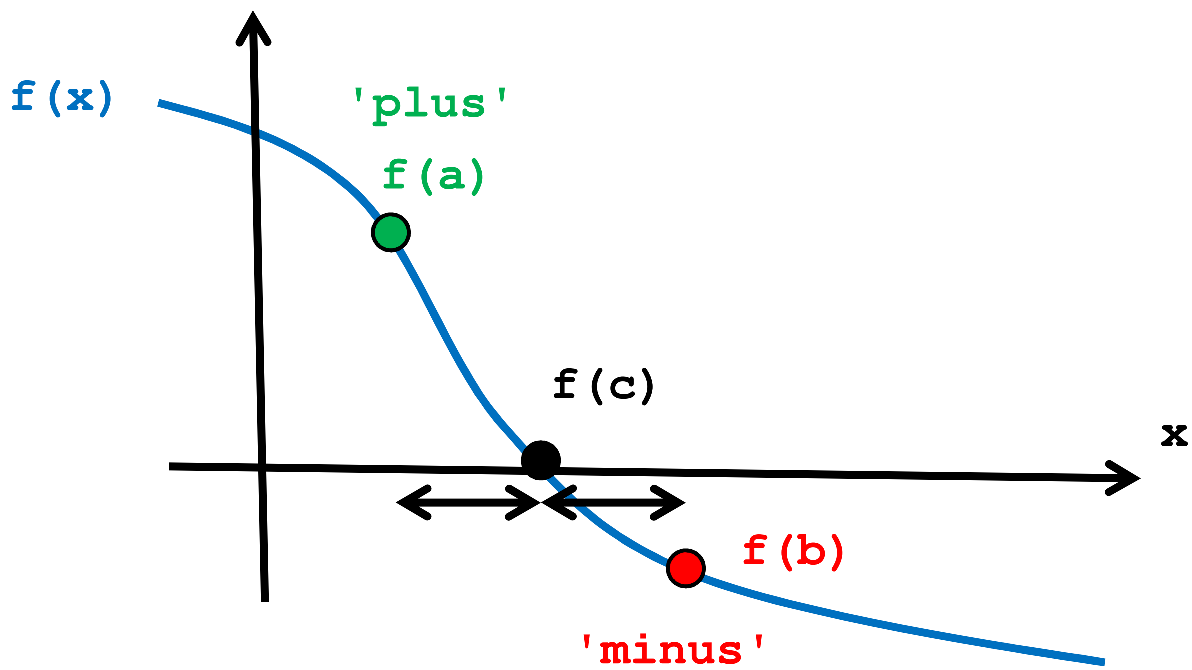
Bisection Method

- evaluate **`f(c)`** where **`c`** is halfway between **`a`** and **`b`**
  - if **`f(c)`** is close enough to zero done



Bisection Method

- otherwise **`c`** becomes the new end point (in this case, `'minus'`) and recursively search the range `'plus'` – `'minus'`



```
public class Bisect {

  // the function we want to find the root of
  public static double f(double x) {
    return Math.cos(x);
  }
```

```
public static double bisect(double xplus, double xminus,
                            double tolerance) {
  // base case
  double c = (xplus + xminus) / 2.0;
  double fc = f(c);
  if( Math.abs(fc) < tolerance ) {
    return c;
  }
  else if (fc < 0.0) {
    return bisect(xplus, c, tolerance);
  }
  else {
    return bisect(c, xminus, tolerance);
  }
}
```

```
  public static void main(String[] args)
  {
        System.out.println("bisection returns: " +
                           bisect(1.0, Math.PI, 0.001));
        System.out.println("true answer      : "
                           + Math.PI / 2.0);
  }
}

prints:

bisection returns: 1.5709519476855602
true answer      : 1.5707963267948966
```

Divide and Conquer

- bisection works by recursively finding which half of the range `'plus'` – `'minus'` the root lies in
  - each recursive call solves the same problem (tries to find the root of the function by guessing at the midpoint of the range)
  - each recursive call solves *one* smaller problem because half of the range is discarded
    - bisection method is decrease and conquer
- divide and conquer algorithms typically recursively divide a problem into several smaller sub-problems until the sub-problems are small enough that they can be solved directly

Recursion (Part 2)

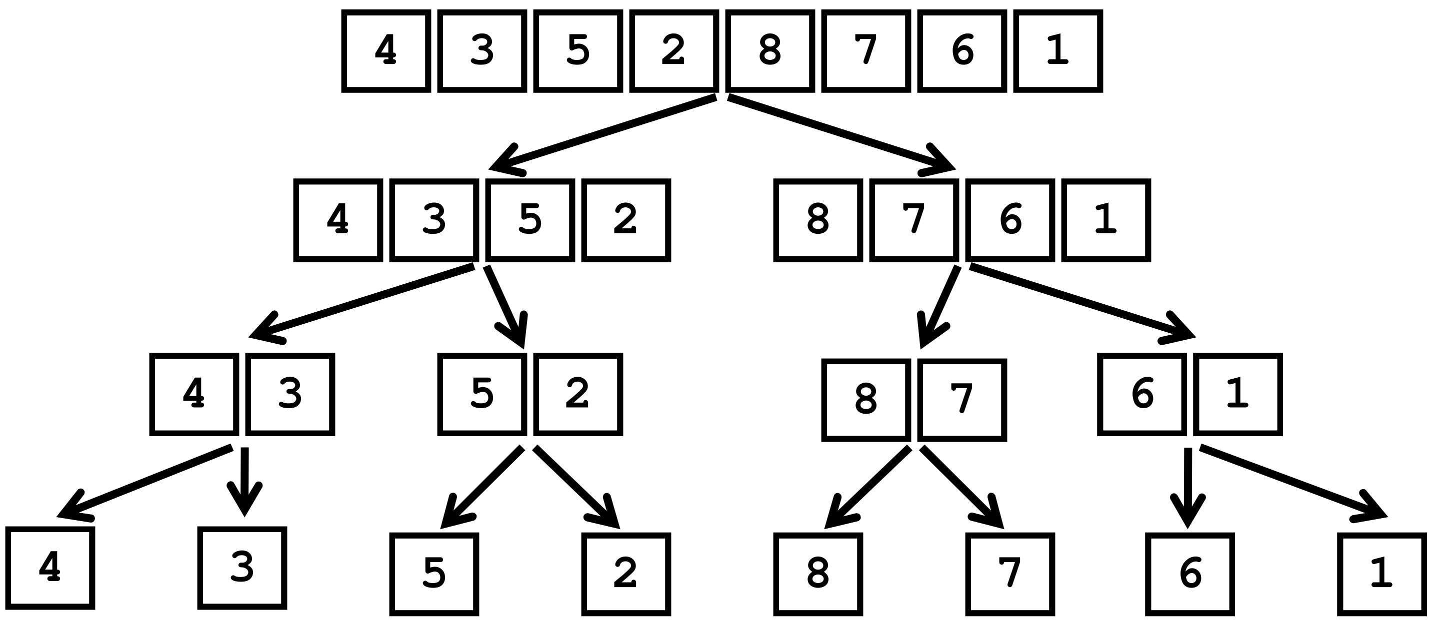
Solving Recurrence Relations

Divide and Conquer

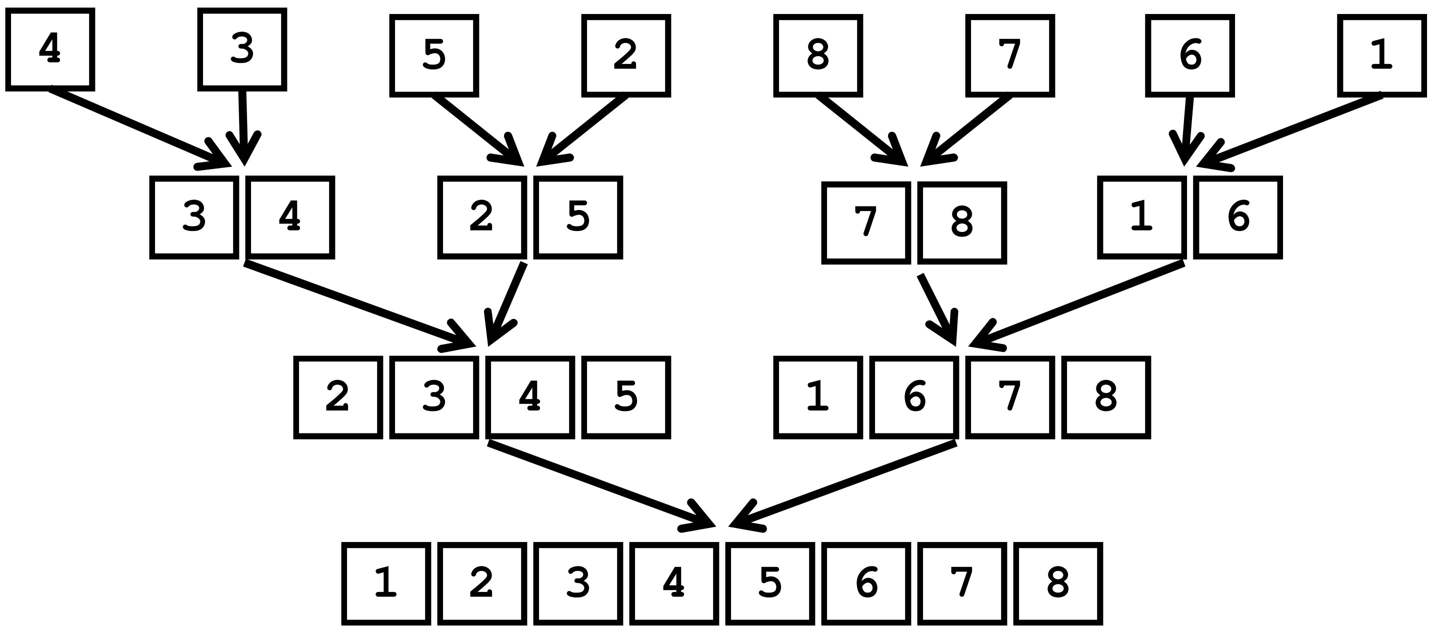
- bisection works by recursively finding which half of the range `'plus'` – `'minus'` the root lies in
  - each recursive call solves the same problem (tries to find the root of the function by guessing at the midpoint of the range)
  - each recursive call solves *one* smaller problem because half of the range is discarded
    - bisection method is decrease and conquer
- divide and conquer algorithms typically recursively divide a problem into several smaller sub-problems until the sub-problems are small enough that they can be solved directly

Merge Sort

- merge sort is a divide and conquer algorithm that sorts a list of numbers by recursively splitting the list into two halves

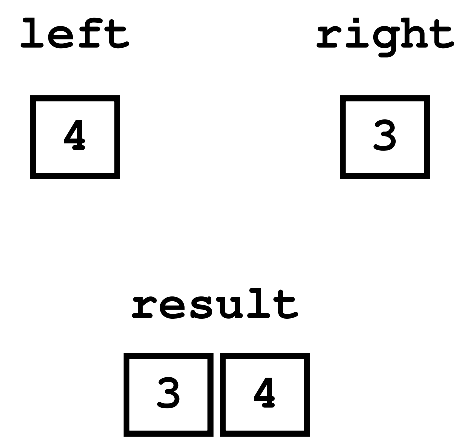


- the split lists are then merged into sorted sub-lists



Merging Sorted Sub-lists

- two sub-lists of length 1



1 comparison
2 copies

```
LinkedList<Integer> result = new LinkedList<Integer>();

int fL = left.getFirst();
int fR = right.getFirst();
if (fL < fR) {
  result.add(fL);
  left.removeFirst();
}
else {
  result.add(fR);
  right.removeFirst();
}
if (left.isEmpty()) {
  result.addAll(right);
}
else {
  result.addAll(left);
}
```

Merging Sorted Sub-lists

- two sub-lists of length 2

`left`

3	4

`right`

2	5

`result`

2	3	4	5

3 comparisons
4 copies

```
LinkedList<Integer> result = new LinkedList<Integer>();

while (left.size() > 0 && right.size() > 0 ) {
  int fL = left.getFirst();
  int fR = right.getFirst();
  if (fL < fR) {
    result.add(fL);
    left.removeFirst();
  }
  else {
    result.add(fR);
    right.removeFirst();
  }
}
if (left.isEmpty()) {
  result.addAll(right);
}
else {
  result.addAll(left);
}
```

Merging Sorted Sub-lists

- two sub-lists of length 4

`left`

2	3	4	5

`right`

1	6	7	8

`result`

1	2	3	4	5	6	7	8

5 comparisons
8 copies

Simplified Complexity Analysis

- in the worst case merging a total of **`n`** elements requires

  `n – 1` comparisons **+**

  `n` copies

  `= 2n – 1` total operations

- we say that the worst-case complexity of merging is the order of *O(n)*
  - *O(…)* is called Big O notation
  - notice that we don't care about the constants 2 and 1

- formally, a function $f(n)$ is an element of $O(n)$ if and only if there is a positive real number M and a real number m such that

$$| f(n) | < Mn \quad \text{for all} \quad n > m$$

- is $2n - 1$ an element of $O(n)$?
  - yes, let $M = 2$ and $m = 0$, then $2n - 1 < 2n$ for all $n > 0$

Informal Analysis of Merge Sort

- suppose the running time (the number of operations) of merge sort is a function of the number of elements to sort
  - let the function be $T(n)$
- merge sort works by splitting the list into two sub-lists (each about half the size of the original list) and sorting the sub-lists
  - this takes $2T(n/2)$ running time
- then the sub-lists are merged
  - this takes $O(n)$ running time
- total running time $T(n) = 2T(n/2) + O(n)$

Solving the Recurrence Relation

$$
\begin{aligned}
T(n) &\rightarrow 2T(n/2) + O(n) \\
&\approx 2T(n/2) + n \\
&= 2[\ 2T(n/4) + n/2\ ] + n \\
&= 4T(n/4) + 2n \\
&= 4[\ 2T(n/8) + n/4\ ] + 2n \\
&= 8T(n/8) + 3n \\
&= 8[\ 2T(n/16) + n/8\ ] + 3n \\
&= 16T(n/16) + 4n \\
&= 2^k T(n/2^k) + kn
\end{aligned}
$$

$T(n)$ approaches...

Solving the Recurrence Relation

$T(n) \quad = \qquad \mathbf{2}^k T(\underline{n/\mathbf{2}^k}) + kn$

- for a list of length $\mathbf{1}$ we know $T(\mathbf{1}) = \mathbf{1}$
  - if we can substitute $T(1)$ into the right-hand side of $T(n)$ we might be able to solve the recurrence

$$\underline{n/\mathbf{2}^k} = \mathbf{1} \;\Rightarrow\; \mathbf{2}^k = n \Rightarrow k = \log(n)$$

Solving the Recurrence Relation

$$
\begin{aligned}
T(n) &= 2^{\log(n)} T(n/2^{\log(n)}) + n \log(n) \\
&= n\, T(1) + n \log(n) \\
&= n + n \log(n) \\
&\in n \log(n)
\end{aligned}
$$

Is Merge Sort Efficient?

- consider a simpler (non-recursive) sorting algorithm called insertion sort

```
// to sort an array a[0]..a[n-1]                    not Java!
for i = 0 to (n-1) {
  k = index of smallest element in sub-array a[i]..a[n-1]
  swap a[i] and a[k]
}
```

```
for i = 0 to (n-1) {                                not Java!
  for j = (i+1) to (n-1) {
    if (a[j] < a[i]) {
      k = j;
    }
  }
  tmp = a[i];   a[i] = a[k];   a[k] = tmp;
}
```

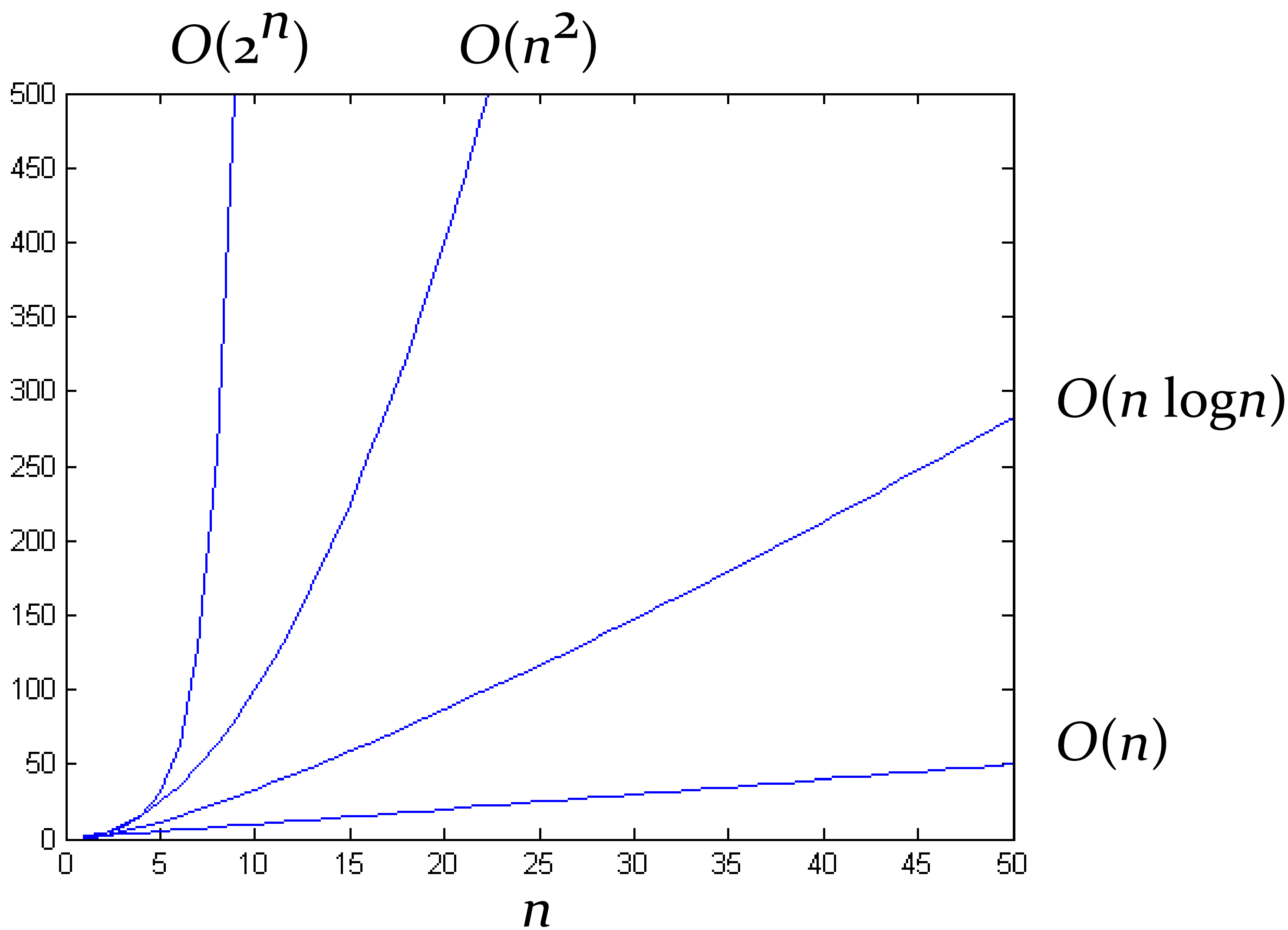
1 comparison + 1 assignment

3 assignments

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{n-1}\left(\left(\sum_{j=(i+1)}^{n-1} 2\right) + 3\right) \\
&= \sum_{i=0}^{n-1}\left(2(n-i-1)\right) \quad + \quad 3n \\
&= 2\sum_{i=0}^{n-1} n \quad - \quad 2\sum_{i=0}^{n-1} i \quad - \quad 2\sum_{i=0}^{n-1} 1 \quad + \quad 3n \\
&= 2n^2 \quad - \quad 2\frac{n(n-1)}{2} \quad - \quad 2n \quad + \quad 3n \\
&= 2n^2 \quad - \quad n^2 \quad + \quad n \quad - \quad 2n \quad + \quad 3n
\end{aligned}
$$

$$
= n^2 \quad + \quad 2n \quad \in \; O(n^2)
$$

Comparing Rates of Growth



Comments

- big O complexity tells you something about the running time of an algorithm as the size of the input, n, approaches infinity
  - we say that it describes the limiting, or asymptotic, running time of an algorithm
- for small values of n it is often the case that a less efficient algorithm (in terms of big O) will run faster than a more efficient one
  - insertion sort is typically faster than merge sort for short lists of numbers

Revisiting the Fibonacci Numbers

- the recursive implementation based on the definition of the Fibonacci numbers is inefficient

```
public static int fibonacci(int n) {
  if (n == 0) {
   return 0;
  }
  else if (n == 1) {
   return 1;
  }
  int f = fibonacci(n - 1) + fibonacci(n - 2);
  return f;
}
```

- how inefficient is it?
- let $T(n)$ be the running time to compute the nth Fibonacci number
  - $T(0) = T(1) = 1$
  - $T(n)$ is a recurrence relation

$$
\begin{aligned}
T(n) \quad &\rightarrow T(n-1)+T(n-2) \\
&= \big(T(n-2)+T(n-3)\big)+T(n-2) \\
&= 2T(n-2)+T(n-3) \\
&> 2T(n-2) \\
&> 2\big(2T(n-4)\big) = 4T(n-4) \\
&> 4\big(2T(n-6)\big) = 8T(n-6) \\
&> 8\big(2T(n-8)\big) = 16T(n-8) \\
&> 2^k T(n-2k)
\end{aligned}
$$

Solving the Recurrence Relation

$T(n) \quad > \qquad 2^k T(\underline{n - 2k})$

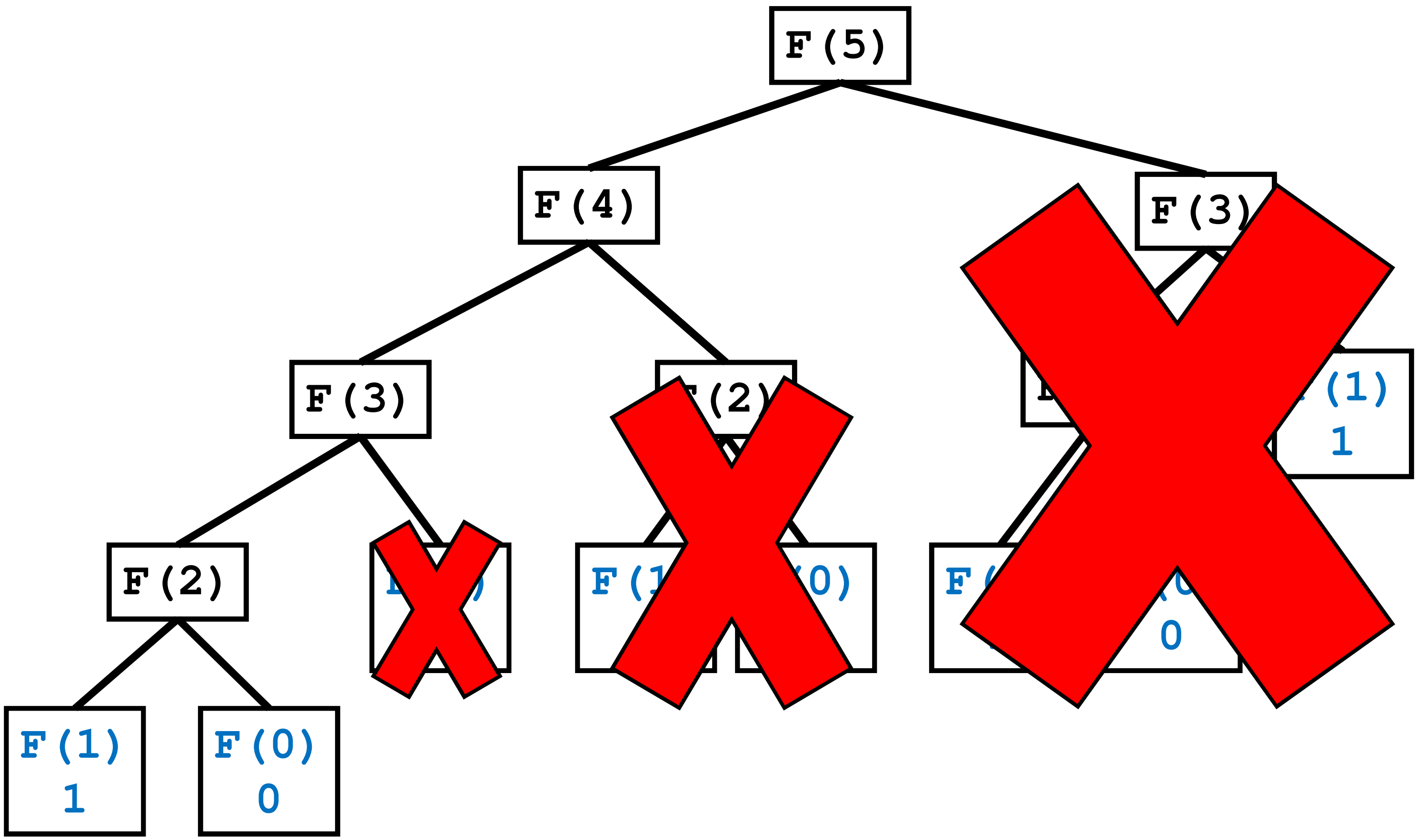
- we know $T(1) = 1$
  - if we can substitute $T(1)$ into the right-hand side of $T(n)$ we might be able to solve the recurrence

$$\underline{n - 2k} = 1 \;\Rightarrow\; 1 + 2k = n \Rightarrow k = (\mathrm{n} - 1)/2$$

$$T(n) > 2^k T(n - 2k) \quad = \quad 2^{(n-1)/2} T(1) \quad = \quad 2^{(n-1)/2} \quad \in \quad O(2^n)$$

An Efficient Fibonacci Algorithm

- an $O(n)$ algorithm exists that computes all of the Fibonacci numbers from $f(0)$ to $f(n)$



- create an array of length $(n + 1)$ and sequentially fill in the array values
  - $O(n)$

```
// pre. n >= 0
public static int[] fibonacci(int n) {
  int[] f = new int[n + 1];
  f[0] = 0;
  f[1] = 1;
  for (int i = 2; i < n + 1; i++) {
    f[i] = f[i - 1] + f[i - 2];
  }
  return f;
}
```

Closing Question

- the recursive Fibonacci and merge sort algorithms can be illustrated using a call tree
  - merge sort is actually 2 trees; one to split and one to merge
- why is the Fibonacci algorithm $O(2^n)$ and merge sort $O(n \log n)$?