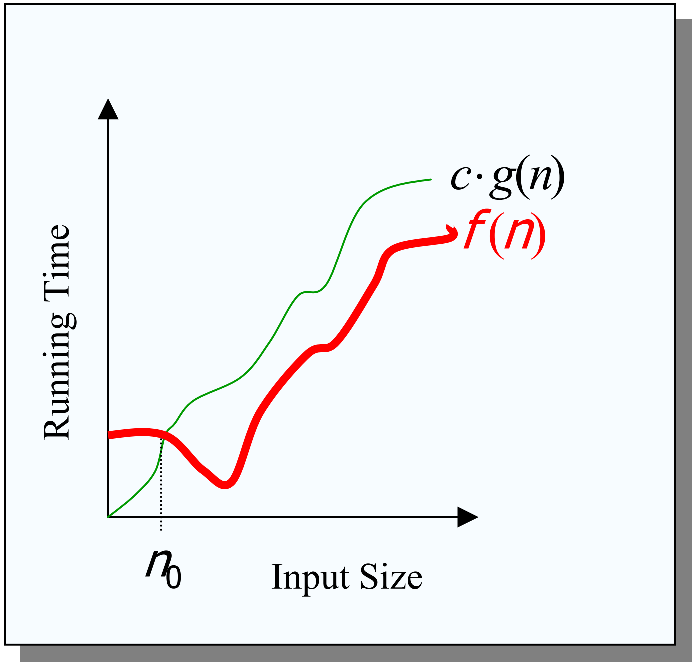
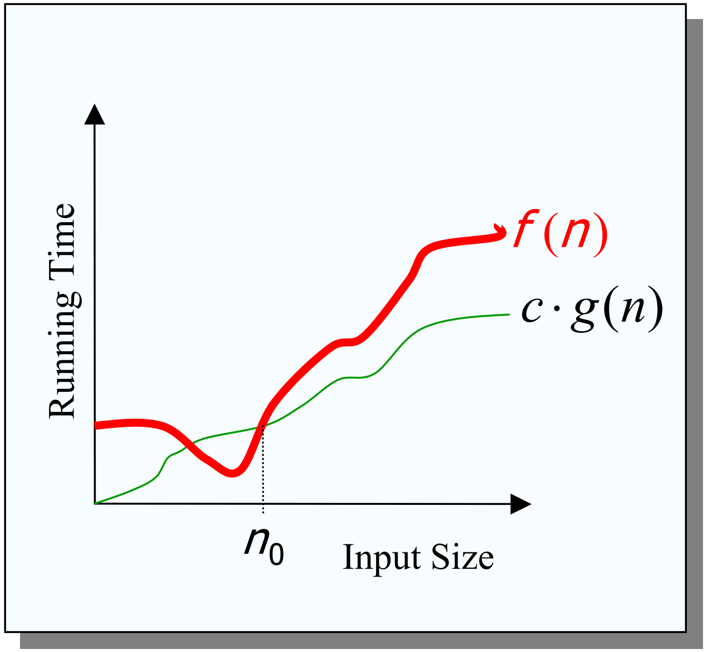
Asymptotic notation

- The "big-Oh" *O*-Notation
  - asymptotic upper bound
  - *f(n)* $\in$ *O(g(n)),* if there exists constants *c* and n_0, $s.t.$ $\mathbf{f(n)} \leq \mathbf{c\ g(n)}$ for $n \geq n_0$
  - *f(n)* and *g(n)* are functions over non-negative integers
- Used for *worst-case* analysis



Asymptotic notation – contd.

- The "big-Omega" Ω–Notation
  - asymptotic lower bound
  - $f(n) \in \Omega(g(n))$ if there exists constants *c* and n_0, *s.t.* $\mathbf{c\ g(n)} \leq \mathbf{f(n)}$ for $n \geq n_0$
- Used to describe *best-case* running times or lower bounds of algorithmic problems
  - E.g., lower-bound of searching in an unsorted array is $\Omega(n)$.



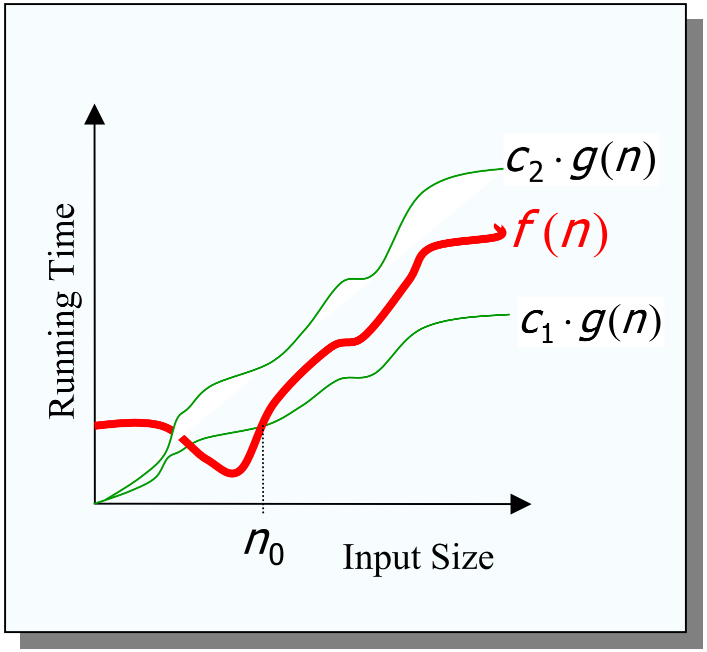
Asymptotic notation – contd.

- Simple Rule: Drop lower order terms and constant factors.
  - $50\, n \log n \in O(n \log n)$
  - $7n - 3 \in O(n)$
  - $8n^2 \log n + 5n^2 + n \in O(n^2 \log n)$

- Note: Even though $50\, n \log n \in O(n^5)$, we usually try to express a O() expression using as small an order as possible

Asymptotic notation – contd.

- The "big-Theta" Θ–Notation
  - asymptoticly tight bound
  - $f(n) \in \Theta(g(n))$ if there exists constants c_1, c_2, and n_0, *s.t.* $\mathbf{c_1\, g(n) \le f(n) \le c_2\, g(n)}$ for $n \ge n_0$
- $f(n) \in \Theta(g(n))$ if and only if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$
- $O(f(n))$ is often misused instead of $\Theta(f(n))$



Proving asymptotic expressions

Use definitions!

e.g. $f(n) = 3n^2 + 7n + 8 = \theta(n^2)$

$f(n) \in \Theta(g(n))$ if there exists constants c_1, c_2, and n_0, *s.t.*

$c_1\, g(n) \le f(n) \le c_2\, g(n)$ for $n \ge n_0$

Here $g(n) = n^2$

One direction $(f(n) = \Omega(g(n))$ is easy

$c_1\, g(n) \le f(n)$ holds for $c_1 = 3$ and $n \ge 0$

The other direction $(f(n) = O(g(n))$ needs more care

$f(n) \le c_2\, g(n)$ holds for $c_2 = 18$ and $n \ge 1$ (CHECK!)

So $n_0 = 1$

Proving asymptotic expressions – contd.

Caveats!

1. constants c_1, c_2 MUST BE POSITIVE.
2. Could have chosen $c_2 = 3 + \varepsilon$ for any $\varepsilon > 0$. WHY?

-- because $7n + 8 \leq \varepsilon n^2$ for $n \geq n_0$ for some sufficiently large n_0. Usually, the smaller the ε you choose, the harder it is to find n_0. So choosing a large ε is easier.

3. Order of quantifiers

$\exists c_1\ c_2\ \exists n_0\ \forall\ n \geq n_0,\ c_1 g(n) \leq f(n) \leq c_2 g(n)$

vs

$\exists n_0\ \forall\ n \geq n_0\ \exists c_1\ c_2,\ c_1 g(n) \leq f(n) \leq c_2 g(n)$

-- allows a different c_1 and c_2 for each n. Can choose $c_2 = 1/n$!! So we can "prove" $n^3 = \Theta(n^2)$.