CSE 2001:
Introduction to Theory of Computation
Summer2013

Week 6: Context-Free Languages

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Course page: http://www.cse.yorku.ca/course/2001

Slides are mostly taken from Suprakash Datta's for Winter 2013

Next

- **Chapter 2:**
  - **Context-Free Languages (CFL)**
  - **Context-Free Grammars (CFG)**
  - **Chomsky Normal Form of CFG**
  - **RL $\subset$ CFL**

Context-Free Languages (Ch. 2)

Context-free languages (CFLs) are a more powerful (augmented) model than FA.

CFLs allow us to describe non-regular languages like $\{ 0^n1^n \mid n \geq 0\}$

General idea: CFLs are languages that can be recognized by automata that have one single stack:

$\{ 0^n1^n \mid n \geq 0\}$ is a CFL

$\{ 0^n1^n0^n \mid n \geq 0\}$ is not a CFL

Context-Free Grammars

Grammars: define/specify a language

Which simple machine produces the non-regular language $\{ 0^n1^n \mid n \in N \}$?

Start symbol S with rewrite rules:

1) $S \rightarrow 0S1$
2) $S \rightarrow$ "stop"

S *yields* 0^n1^n according to

$S \rightarrow 0S1 \rightarrow 00S11 \rightarrow \ldots \rightarrow 0^nS1^n \rightarrow 0^n1^n$

Context-Free Grammars (Def.)

A <u>context free grammar</u> $G=(V,\Sigma,R,S)$ is defined by

- V: a finite set <u>variables</u>
- Σ: finite set <u>terminals</u> (with $V \cap \Sigma = \emptyset$)
- R: finite set of <u>substitution rules</u> $V \rightarrow (V \cup \Sigma)^*$
- S: <u>start symbol</u> $\in V$

The <u>language of grammar</u> G is denoted by L(G):

$$L(G) = \{ w \in \Sigma^* \mid S \Rightarrow^* w \}$$

Derivation $\Rightarrow^*$

A single step derivation "$\Rightarrow$" consist of the substitution of a variable by a string according to a substitution rule.

Example: with the rule "$A \rightarrow BB$", we can have the derivation "$01AB0 \Rightarrow 01BBB0$".

A sequence of several derivations (or none) is indicated by "$\Rightarrow^*$"
Same example: "$0AA \Rightarrow^* 0BBBB$"

Some Remarks

The language $L(G) = \{ w \in \Sigma^* \mid S \Rightarrow^* w \}$ contains only strings of terminals, not variables.

Notation: we summarize several rules, like

$A \rightarrow B$
$A \rightarrow 01$ by $A \rightarrow B \mid 01 \mid AA$
$A \rightarrow AA$

Unless stated otherwise: topmost rule concerns the start variable

Context-Free Grammars (Ex.)

Consider the CFG $G=(V,\Sigma,R,S)$ with

$V = \{S\}$
$\Sigma = \{0,1\}$
R: $S \rightarrow 0S1 \mid 0Z1$
$Z \rightarrow 0Z \mid \varepsilon$

Then $L(G) = \{0^i 1^j \mid i \geq j \}$

S <u>yields</u> $0^{j+k}1^j$ according to:

$S \Rightarrow 0S1 \Rightarrow \ldots \Rightarrow 0^j S 1^j \Rightarrow 0^j Z 1^j \Rightarrow 0^j 0 Z 1^j \Rightarrow$
$\ldots \Rightarrow 0^{j+k} Z 1^j \Rightarrow 0^{j+k} \varepsilon 1^j = 0^{j+k} 1^j$

Importance of CFL

Model for natural languages (Noam Chomsky)

Specification of programming languages:
"parsing of a computer program"

Describes mathematical structures

Intermediate between regular languages and computable languages (Chapters 3,4,5 and 6)

Example Boolean Algebra

Consider the CFG $G=(V,\Sigma,R,S)$ with
$V = \{S,Z\}$
$\Sigma = \{0,1,(,),\neg,\vee,\wedge\}$
R: $S \rightarrow 0 \mid 1 \mid \neg(S) \mid (S)\vee(S) \mid (S)\wedge(S)$

Some elements of L(G):

0
$\neg((\neg(0))\vee(1))$
$(1)\vee((0)\wedge(0))$

Note: Parentheses prevent "$1\vee0\wedge0$" confusion.

Human Languages

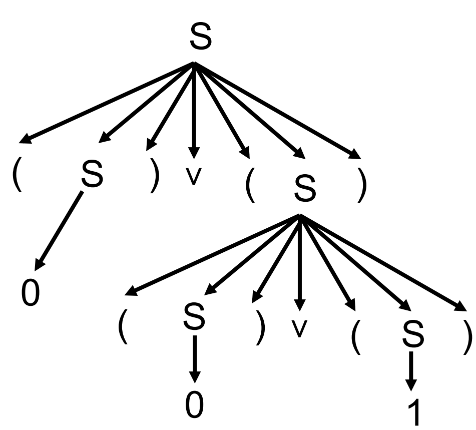
Number of rules:

<SENTENCE> → <NOUN-PHRASE><VERB-PHRASE>
<NOUN-PHRASE> → <CMPLX-NOUN> | <CMPLX-NOUN><PREP-PHRASE>
<VERB-PHRASE> → <CMPLX-VERB> | <CMPLX-VERB><PREP-PHRASE>
<CMPLX-NOUN> → <ARTICLE><NOUN>
<CMPLX-VERB> → <VERB> | <VERB><NOUN-PHRASE> …
<ARTICLE> → a | the
<NOUN> → boy | girl | house
<VERB> → sees | ignores

Possible element: `the boy sees the girl`

Parse Trees

The parse tree of (0)∨((0)∧(1)) via rule
S → 0 | 1 | ¬(S) | (S)∨(S) | (S)∧(S):



Ambiguity

A grammar is <u>ambiguous</u> if some strings are derived <u>ambiguously</u>.

A string is derived <u>ambiguously</u> if it has more than one <u>leftmost derivations</u>.

Typical example: rule $S \rightarrow 0 \mid 1 \mid S{+}S \mid S{\times}S$

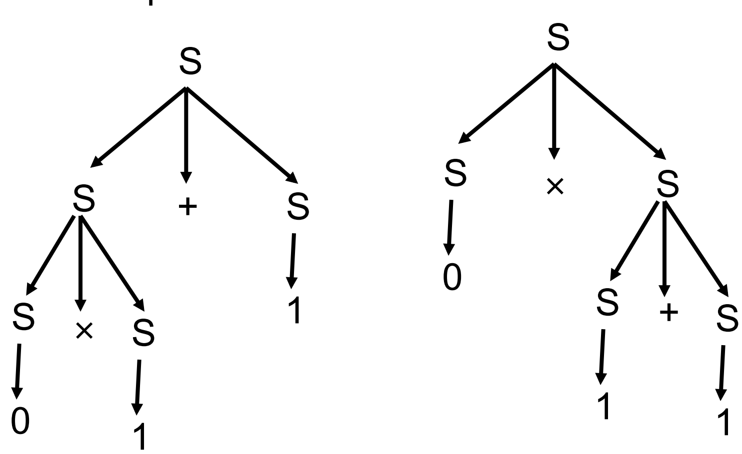
$S \Rightarrow S{+}S \Rightarrow S{\times}S{+}S \Rightarrow 0{\times}S{+}S \Rightarrow 0{\times}1{+}S \Rightarrow 0{\times}1{+}1$

versus

$S \Rightarrow S{\times}S \Rightarrow 0{\times}S \Rightarrow 0{\times}S{+}S \Rightarrow 0{\times}1{+}S \Rightarrow 0{\times}1{+}1$

Ambiguity and Parse Trees

The ambiguity of $0{\times}1{+}1$ is shown by the two different parse trees:



More on Ambiguity

The two different derivations:
S ⇒ S+S ⇒ 0+S ⇒ 0+1
and
S ⇒ S+S ⇒ S+1 ⇒ 0+1
do *not* constitute an ambiguous string 0+1
(they will have the same parse tree)

Languages that can only be generated by ambiguous grammars are "inherently ambiguous"

Context-Free Languages

Any language that can be generated by a context free grammar is a context-free language (CFL).

The CFL $\{ 0^n 1^n \mid n \geq 0 \}$ shows us that certain CFLs are nonregular languages.

Q1: Are all regular languages context free?

Q2: Which languages are outside the class CFL?

"Chomsky Normal Form"

A context-free grammar G = (V,Σ,R,S) is in Chomsky normal form if every rule is of the form

$A \rightarrow BC$

or $A \rightarrow x$

with variables $A \in V$ and $B,C \in V \setminus \{S\}$, and $x \in \Sigma$

For the start variable S we also allow the rule

$S \rightarrow \varepsilon$

Advantage: Grammars in this form are far easier to analyze.

Theorem 2.9

Every context-free language can be described by a grammar in Chomsky normal form.

Outline of Proof:
We rewrite every CFG in Chomsky normal form.
We do this by replacing, one-by-one, every rule that is not 'Chomsky'.
We have to take care of: Starting Symbol, ε symbol, all other violating rules.

Proof of Theorem 2.9

Given a context-free grammar $G = (V,\Sigma,R,S)$, rewrite it to Chomsky Normal Form by

1) New start symbol S_0 (and add rule $S_0 \to S$)
2) Remove $A \to \varepsilon$ rules (*from the tail*): before: $B \to xAy$ and $A \to \varepsilon$, after: $B \to xAy \mid xy$
3) Remove unit rules $A \to B$ (*by the head*): "$A \to B$" and "$B \to xCy$", becomes "$A \to xCy$" and "$B \to xCy$"
4) Shorten all rules to two: before: "$A \to B_1B_2\ldots B_k$", after: $A \to B_1A_1$, $A_1 \to B_2A_2, \ldots, A_{k-2} \to B_{k-1}B_k$
5) Replace ill-placed terminals "a" by T_a with $T_a \to a$

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Careful Removing of Rules

Do not introduce new rules that you removed earlier.

Example: $A \rightarrow A$ simply disappears

When removing $A \rightarrow \varepsilon$ rules, insert *all* new replacements:

$B \rightarrow AaA$ becomes $B \rightarrow AaA \mid aA \mid Aa \mid a$

Example of Chomsky NF

Initial grammar: $S \rightarrow aSb \mid \varepsilon$
In Chomsky normal form:

$S_0 \rightarrow \varepsilon \mid T_aT_b \mid T_aX$
$X \rightarrow ST_b$
$S \rightarrow T_aT_b \mid T_aX$
$T_a \rightarrow a$
$T_b \rightarrow b$

RL ⊆ CFL

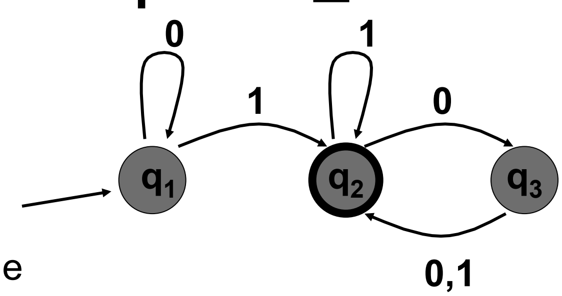
Every regular language can be expressed by a context-free grammar.

Proof Idea:
Given a DFA $M = (Q,\Sigma,\delta,q_0,F)$, we construct a corresponding CF grammar $G_M = (V,\Sigma,R,S)$ with $V = Q$ and $S = q_0$
Rules of G_M:

$q_i \to x\,\delta(q_i,x)$ for all $q_i \in V$ and all $x \in \Sigma$
$q_i \to \varepsilon$ for all $q_i \in F$

Example RL ⊆ CFL

The DFA



leads to the context-free grammar $G_M = (Q,\Sigma,R,q_1)$ with the rules

$q_1 \to 0q_1 \mid 1q_2$
$q_2 \to 0q_3 \mid 1q_2 \mid \varepsilon$
$q_3 \to 0q_2 \mid 1q_2$

Picture Thus Far