

Assignment 1 – Solutions

Total marks: 90.

Out: May 20

Due: June 3 by 18:45 in the dropbox, or 19:00 in class

Note that:

- The assignment can be handwritten or typed. It **must** be legible.
- You must do this assignment individually.
- Submit this assignment only if you have read and understood the policy on academic honesty on the course web page. If you have questions or concerns, please contact the instructor.
- Use the dropbox near the main office to submit your assignments, or hand them in at the beginning of class (please note the times and day above). No late submissions will be accepted.

1. [15 points, 5 points each] Write each of the following sets explicitly:

a) $\{a\} \times \{a, b\} \times \{a, b, c\}$

Solution: $\{(a, a, a), (a, a, b), (a, a, c), (a, b, a), (a, b, b), (a, b, c)\}$

b) $\emptyset \times \{a, b, c\}$

Solution: \emptyset

c) $\mathcal{P}(\{a, b\}) \times \{a, b\}$

Solution:

$$\{(\emptyset, a), (\emptyset, b), (\{a\}, a), (\{b\}, a), (\{a\}, b), (\{b\}, b), (\{a, b\}, a), (\{a, b\}, b)\}$$

2. [10 points, 5 points each] Give a recursive definition for each of the following sets:

a) $\{m \mid m = 5k + 1, k \in \mathcal{N}\}$

Solution: The least set S such that $6 \in S$ and if $x \in S$ then $x + 5 \in S$. (You may also include 1 in S . The textbook understands \mathcal{N} as the positive integers, but usually \mathcal{N} stands for the natural numbers including 0.)

b) $\{a^{3n} \mid n \in \mathcal{N}\}$

Solution: The least set S such that $a^3 \in S$ and if $x \in S$ then $x \cdot a^3 \in S$. (You may also include 1 in S for the reason mentioned above.)

3. [15 points, 5 points each] Let the predicate $S(x)$ represent the statement “ x is a student”, $C(y)$ represent “ y is a course”, and $L(x, y)$ represent “ x likes y ”. Write sentences in predicate logic that represent the following English statements:

a) There is a course that no student likes.

Solution: $\exists c(C(c) \wedge \neg \exists s(S(s) \wedge L(s, c)))$ or
 $\exists c(C(c) \wedge \forall s(S(s) \rightarrow \neg L(s, c)))$

b) Every student likes some course.

Solution: $\forall s(S(s) \rightarrow \exists c(C(c) \wedge L(s, c)))$

c) Some student dislikes two distinct courses.

Solution: $\exists s \exists c_1 \exists c_2 (S(s) \wedge C(c_1) \wedge C(c_2) \wedge c_1 \neq c_2 \wedge \neg L(s, c_1) \wedge \neg L(s, c_2))$

4. [10 points] Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution:

$x \in A \cup (B \cap C)$

iff $x \in A$, or $x \in B$ and $x \in C$

iff both $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

iff $x \in (A \cup B) \cap (A \cup C)$

5. [10 points] Prove by induction on $|A|$ that for any finite set A , $|\mathcal{P}(A)| = 2^{|A|}$.

Solution:

Base case: For $|A| = 0$. Then $A = \emptyset$. Then $\mathcal{P}(A) = \{\emptyset\}$. Thus $|\mathcal{P}(A)| = 1 = 2^{|A|}$, and the result holds.

Inductive step: Assume that the result holds for all sets A such that $|A| = n$, for some natural number n (induction hypothesis). Show that it must also hold for all sets B such that $|B| = n + 1$. Take an arbitrary set B such that $|B| = n + 1$. Then B cannot be empty. Let $x \in B$. Then $|B \setminus \{x\}| = n$. By the induction hypothesis $|\mathcal{P}(B \setminus \{x\})| = 2^n$. $\mathcal{P}(B) = \mathcal{P}(B \setminus \{x\}) \cup \{s \cup \{x\} \mid s \in \mathcal{P}(B \setminus \{x\})\}$. Thus $|\mathcal{P}(B)| = 2^n + 2^n = 2 \cdot 2^n = 2^{n+1} = 2^{|B|}$. i.e., the result holds for B .

Therefore the result holds for all natural numbers n , which covers all finite sets.

6. [10 points] Let M be a deterministic finite automaton. Under exactly what circumstances is $\epsilon \in L(M)$ (where ϵ denotes the empty string)? Prove your answer.

Solution:

Let $M = (Q, \Sigma, \delta, q_0, F)$ where q_0 is the initial state and F is the set of accepting states. Then:

$\epsilon \in L(M)$ iff $q_0 \in F$ (4 points)

Proof: (\leftarrow) (3 points) If $q_0 \in F$, then M accepts ϵ as the sequence of states q_0 satisfies the conditions in the definition of accepting computation for ϵ , i.e. q_0 is the initial

state and it is an accepting state.

(\rightarrow) (3 points) We show the contrapositive, i.e., if $q_0 \notin F$ then $\epsilon \notin L(M)$. Assume $q_0 \notin F$. Then the sequence of states q_0 does not satisfy the third condition for accepting computations as $q_0 \notin F$. Since ϵ is the empty string, no sequence of states q_0, q_1, \dots is an accepting computation for ϵ as there is no first character w_1 in ϵ and the second condition $\delta(q_0, w_1) = q_1$ is violated.

7. [20 points, 10 points each] Define formally deterministic finite automata accepting each of the following languages. Also provide an English description of what each state represents.

- a) $\{w \in \{c, d\} \mid w \text{ has } dccd \text{ as a substring}\}$ (should be $\{w \in \{c, d\}^* \mid w \text{ has } dccd \text{ as a substring}\}$)

Solution:

This language is recognized by $M = (Q, \Sigma, \delta, q_0, F)$ where

$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$

$$\Sigma = \{c, d\}$$

$$F = \{q_4\}$$

$$\delta(q_0, c) = q_0 \quad \delta(q_0, d) = q_1$$

$$\delta(q_1, c) = q_2 \quad \delta(q_1, d) = q_1$$

$$\delta(q_2, c) = q_3 \quad \delta(q_2, d) = q_1$$

$$\delta(q_3, c) = q_0 \quad \delta(q_3, d) = q_4$$

$$\delta(q_4, c) = q_4 \quad \delta(q_4, d) = q_4$$

q_0 represents the situation where no part of $dccd$ has yet been scanned,
 q_1 represents the situation where only the first d in $dccd$ has been scanned,
 q_2 represents the situation where the first d and c in $dccd$ have been scanned,
 q_3 represents the situation where dcc has been scanned, and
 q_4 represents the situation where the whole target substring $dccd$ has been scanned.

- b) $\{w \in \{c, d\} \mid w \text{ has an odd number of } c\text{'s and an even number of } d\text{'s}\}$ (should be $\{w \in \{c, d\}^* \mid w \text{ has an odd number of } c\text{'s and an even number of } d\text{'s}\}$)

Solution:

This language is recognized by $M = (Q, \Sigma, \delta, q_0, F)$ where

$$Q = \{q_0, q_1, q_2, q_3\}$$

$$\Sigma = \{c, d\}$$

$$F = \{q_1\}$$

$$\delta(q_0, c) = q_1 \quad \delta(q_0, d) = q_2$$

$$\delta(q_1, c) = q_0 \quad \delta(q_1, d) = q_3$$

$$\delta(q_2, c) = q_3 \quad \delta(q_2, d) = q_0$$

$$\delta(q_3, c) = q_2 \quad \delta(q_3, d) = q_1$$

q_0 represents the situation where an even number of c 's and an even number of d 's have been scanned,

q_1 represents the situation where an odd number of c 's and an even number of d 's have been scanned,

q_2 represents the situation where an even number of c 's and an odd number of d 's have been scanned,

q_3 represents the situation where an odd number of c 's and an odd number of d 's have been scanned.