Homework Assignment #7
Due: December 3, 2012

1. A context-free grammar (CFG) is defined by

- a finite alphabet V of variables,
- a finite alphabet T of terminals,
- a start symbol $S \in V$, and
- a finite set of rules R, where each rule is of the form $A \to \alpha$ with $A \in V$ and $\alpha \in (V \cup T)^*$.

We say that a string $x \in T^*$ is generated by the grammar if there is a sequence $\beta_0, \beta_1, \beta_2, \ldots, \beta_k$ such that

- for all i, $\beta_i \in (V \cup T)^*$,
- $\beta_0 = S$,
- $\beta_k = x$, and
- for all i, there exists a rule $A \to \alpha$ such that β_{i+1} can be obtained from β_i by replacing one occurrence of A in β_i by α.

(Notice that strings generated by the grammar contain only terminals and no variables.)

Prove that the problem of determining whether a given CFG generates at least one string is **P**-complete.

Hint: use a reduction involving the monotone circuit value problem. Each variable of the CFG will correspond to a node of the circuit.

2. We will discuss the following probabilistic complexity classes in the November 28 lecture.

A language L is in **RP** iff there is a probabilistic algorithm A that always terminates within a polynomial number of steps and

- for all $x \in L$, $A(x)$ outputs yes with probability at least $\frac{1}{2}$ and
- for all $x \notin L$, $A(x)$ outputs yes with probability 0.

A language L is in **BPP** iff there is a probabilistic algorithm A that always terminates within a polynomial number of steps and

- for all $x \in L$, $A(x)$ outputs yes with probability at least $\frac{3}{4}$ and
- for all $x \notin L$, $A(x)$ outputs yes with probability at most $\frac{1}{4}$.

We shall see in class that it is trivial to prove **RP** $\subseteq$ **NP**.

(a) Suppose you had a algorithm $DecideSAT(\phi)$ that solves SAT (i.e., it decides whether formula ϕ is satisfiable). Show how you could construct a satisfying truth value assignment for ϕ, if one exists, by calling $DecideSAT$ $O(n)$ times, where n is the number of variables that appear in ϕ.

(b) Prove that if SAT is in **BPP**, then it is in **RP**.

You may use the following fact without proving it: $\lim_{x\to\infty}(1-\frac{1}{x})^x = \frac{1}{e}$.

(c) Prove that if **NP** $\subseteq$ **BPP** then **NP** = **RP**