CSE 2001 Fall 2012 Second test - version 2
Solutions

1. (3 points) Prove that the following language is not regular using the Pumping Lemma. You must include all the steps of the proof to get full credit. Assume that $\Sigma = \{0, 1\}$.

$$\{0^{2n}1^{2n} | n \geq 0\}$$

Solution: Assume that the given language (call it L) is regular. Therefore there exists a DFA for it. Let p be the pumping length. We choose string $s = a^{2p}b^{2p} = xyz$ for pumping. From the third condition of the Pumping Lemma, we know that $|xy| \leq p$ so y consists of only a's. Pumping down we get the string $s' = a^{2p-k}b^{2p} \in L$ where $k > 0$. This is a contradiction because s' does not satisfy the criterion in the definition of L. Therefore L is not regular.

2. (3 points) Assume that $\Sigma = \{a, b\}$. Is the following language context-free? First write yes or no and then provide a CFG/PDA if it is context-free or a Pumping Lemma proof if it is not.

$$\{a^m b^n a^n b^m | m \geq 0, n \geq 0\}$$

Solution: The language is context-free. A CFG for this language is

1. $S \rightarrow aSb|U$
2. $U \rightarrow bUa|\epsilon$

The variable S generates strings of the form $a^k b^k$ and then inserts in the centre the word produced by the variable U. The variable U generates strings of the form $b^j a^j$.

3. (2 points) Define the Chomsky normal form for context-free languages.

Solution: The criteria for CNF are as follows (from page 109, third edition of Sipser):

1. Every rule is of the form $A \rightarrow BC$ or $A \rightarrow a$, where A is any variable, B, C are any variables that are not the start variable and a is a terminal.
2. The only rule that can have an ϵ on the right is $S \rightarrow \epsilon$.

4. (3 points) Prove that context-free languages are closed under concatenation.

Solution: Given two context-free languages $L_1 = (V_1, \Sigma_1, R_1, S_1)$ and $L_2 = (V_2, \Sigma_2, R_2, S_2)$, we can construct $L = L_1 \cdot L_2$ as $L = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S \rightarrow S_1 S_2\}, S)$.

The above construction uses a new start state and an extra rule $S \rightarrow S_1 S_2$. Thus every string $s = uv \in L, u \in L_1, v \in L_2$ is in L because u is derivable from S_1 and v is derivable from S_2.

5. (3 points) Assume that $\Sigma = \{a, b\}$. Construct a CFG for the following language $\{a^i b^j | 2i = 3j\}$. Give a very brief explanation of why it is correct.

Solution: Note that j cannot be odd, because if it is then $3j$ is odd. Listing the first few values of j, we see that the corresponding strings are $\epsilon, aaabb, a^3b^2, a^6b^4$ and so on. The general formula is $a^{3k}b^{2k}$. So the CFG is
$S \rightarrow \epsilon | aaaSbb$.

6. (3 points) Assume that $\Sigma = \{a\}$. Prove that the following language is not context-free using the Pumping Lemma. You must include all the steps of the proof to get full credit.

$$L = \{a^{n^2} | n \geq 0\}$$

Solution: Assume that the given language L is context-free. Therefore there exists a PDA for it. Let p be the pumping length. We choose string $s = a^{p^2} = uvxyz$ for pumping. From the third condition we know that $|vxy| \leq p$. Obviously vxy consists of only a's. Suppose $|vy| = k$. Pumping up once we get the string $s' = a^{p^2+k} \in L$ where $p \geq k > 0$. We see that s' does not satisfy the criterion in the definition of L – observe that $p^2 + k > p^2$ and $p^2 + k < (p + 1)^2$ since $k > 0$. Thus $p^2 + k$ cannot be a perfect square. This is a contradiction. Therefore L is not context-free.

7. (3 points) Let us denote by w^R the reverse of a string w. Similarly denote by L^R the language consisting of all strings of L reversed. In class, it was proved that the language $PAL = \{ww^R | w \in \Sigma^*\}$ is not regular. Point out the major flaw in the following (erroneous) proof. You must explain why one of the statements is incorrect, and not just refer to the statement.

    1. If L is a regular language, so is L^R.
    2. The concatenation of two regular languages is regular.
    3. PAL is the concatenation of L and L^R.
    4. Therefore PAL is regular.

    Solution: Step 3 is the erroneous step.

    Note that $L \cdot L^R = \{wv^R | w \in L, v \in L\}$. So, $PAL \subset L \cdot L$, but $PAL \neq L \cdot L$.