CSE 2001:
Introduction to Theory of Computation

Fall 2012

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Course page: http://www.cs.yorku.ca/course/2001

Administrivia

Lectures: Tue, Thu 4 - 5:30 pm (CLH J)

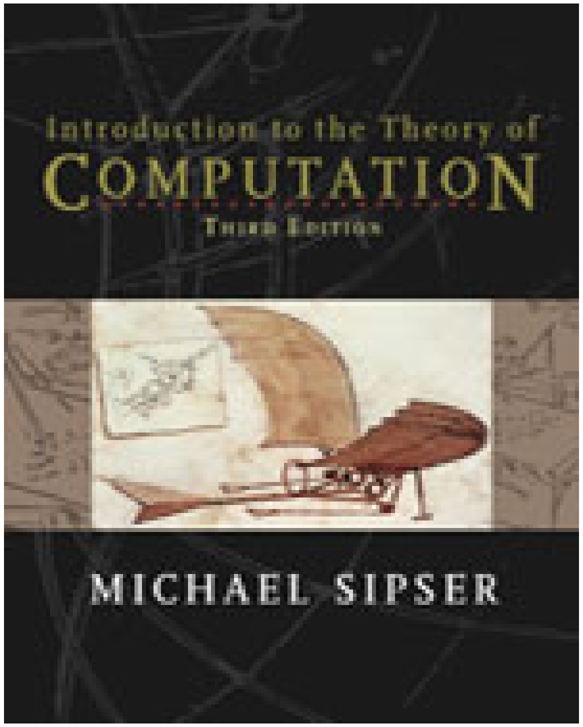
Office hours: Tues 5:30-7 pm, Wed 4-6 pm (CSEB 3043), or by appointment.

TA: Paria Mehrani, will lead problem-solving sessions. Another grader will only grade some homework, no office hours.

http://www.cs.yorku.ca/course/2001

Webpage: All announcements/handouts will be published on the webpage -- check often for updates)

Textbook:



Michael Sipser. *Introduction to the Theory of Computation, Third Edition*. Cengage Learning, 2013.

Administrivia – contd.

Grading:
2 Midterms : 20% + 20% (in class)
Final: 40%
Assignments (4 sets): 20%

Grades will be on *ePost (linked from the web page)*

Notes:

1. All assignments are individual.
2. There MAY be 1-2 extra credit quizzes. These will be announced beforehand.

Administrivia – contd.

Plagiarism: Will be dealt with very strictly. Read the detailed policies on the webpage.

Handouts (including solutions): in /cs/course/2001

Slides: Will usually be on the web the morning of the class. The slides are for MY convenience and for helping you recollect the material covered. They are not a substitute for, or a comprehensive summary of, the textbook.

Resources: We will follow the textbook closely.

There are more resources than you can possibly read – including books, lecture slides and notes.

Recommended strategy

- This is an applied Mathematics course -- practice instead of reading.
- Try to get as much as possible from the lectures.
- If you need help, get in touch with me early.
- If at all possible, try to come to the class with a fresh mind.
- Keep the big picture in mind. ALWAYS.

Course objectives - 1

Reasoning about computation

- Different computation models
  - Finite Automata
  - Pushdown Automata
  - Turing Machines
- What these models can and cannot do

Course objectives - 2

- What does it mean to say “there does not exist an algorithm for this problem”?
- Reason about the hardness of problems
- Eventually, build up a hierarchy of problems based on their hardness.

Course objectives - 3

- We are concerned with solvability, NOT efficiency.
- CSE 3101 (Design and Analysis of Algorithms) efficiency issues.

Reasoning about Computation

Computational problems may be

- Solvable, quickly
- Solvable in principle, but takes an infeasible amount of time (e.g. thousands of years on the fastest computers available)
- (provably) not solvable

Theory of Computation: parts

- Automata Theory (CSE 2001)
- Complexity Theory (CSE 3101, 4115)
- Computability Theory (CSE 2001, 4101)

Reasoning about Computation - 2

- Need formal reasoning to make credible conclusions
- Mathematics is the language developed for formal reasoning
- As far as possible, we want our reasoning to be intuitive

Next:

Ch. 0:Set notation and languages

- Sets and sequences
- Tuples
- Functions and relations
- Graphs
- Boolean logic: $\vee \;\; \wedge \;\; \neg \;\; \Leftrightarrow \;\; \Rightarrow$

- **Review of proof techniques**
  - Construction, Contradiction, Induction…

Some of these slides are adapted from Wim van Dam's slides (www.cs.berkeley.edu/~vandam/CS172/) and from Nathaly Verwaal (http://cpsc.ucalgary.ca/~verwaal/313/F2005)

Topics you should know:

- Elementary set theory
- Elementary logic
- Functions
- Graphs

Set Theory review

- Definition
- Notation: $A = \{ x \mid x \in \mathbf{N} , x \bmod 3 = 1\}$
  $\mathbf{N} = \{1,2,3,\ldots\}$
- Union: $A \cup B$
- Intersection: $A \cap B$
- Complement: $\overline{A}$
- Cardinality: $|A|$
- Cartesian Product:

$$A \times B = \{ (x,y) \mid x \in A \text{ and } y \in B\}$$

Some Examples

$L_{<6} = \{ x \mid x \in \mathbf{N} , x<6 \}$

$L_{prime} = \{x \mid x \in \mathbf{N}, x \text{ is prime}\}$

$L_{<6} \cap L_{prime} = \{2,3,5\}$

$\Sigma = \{0,1\}$

$\Sigma\times\Sigma= \{(0,0), (0,1), (1,0), (1,1)\}$

Formal: $A\cap B = \{ x \mid x\in A \text{ and } x\in B\}$

Power set

"Set of all subsets"

Formal: $\mathcal{P}(A) = \{ S \mid S \subseteq A \}$

Example: $A = \{x,y\}$

$\mathcal{P}(A) = \{ \{\} , \{x\} , \{y\} , \{x,y\} \}$

Note the different sizes: for finite sets

$$|\mathcal{P}(A)| = 2^{|A|}$$

$$|A \times A| = |A|^2$$

Graphs: review

- Nodes, edges, weights
- Undirected, directed
- Cycles, trees
- Connected

Logic: review

Boolean logic: $\vee \;\wedge\; \neg$

Quantifiers: $\forall$, $\exists$

<u>statement</u>: Suppose $x \in \mathbf{N}$, $y \in \mathbf{N}$.

Then $\boxed{\forall x \; \exists y \; y > x}$

for any integer, there exists a larger integer

$\Rightarrow$: $a \Rightarrow b$ "is the same as" (is logically equivalent to) $\neg a \vee b$

$\Leftrightarrow$: $a \Leftrightarrow b$ is logically equivalent to

$(a \Rightarrow b) \wedge (b \Rightarrow a)$

Logic: review - 2

Contrapositive and converse:

the converse of $a \Rightarrow b$ is $b \Rightarrow a$

the contrapositive of $a \Rightarrow b$ is $\neg b \Rightarrow \neg a$

Any statement is logically equivalent to its contrapositive, but not to its converse.

Logic: review - 3

Negation of statements

$\neg(\forall x\ \exists y\ y > x)$ "=" $\exists x\ \forall y\ y \le x$

(LHS: negation of "for any integer, there exists a larger integer", RHS: there exists a largest integer)

TRY: $\neg(a \Rightarrow b) = ?$

Logic: review - 4

Understand quantifiers

$\forall x\ \exists y\ P(y, x)$ is not the same as

$\exists y\ \forall x\ P(y, x)$

Consider $P(y,x)$: $x \leq y$.

$\forall x\ \exists y\ x \leq y$ is TRUE over **N** (set $y = x + 1$)

$\exists y\ \forall x\ x \leq y$ is FALSE over **N** (there is no largest number in **N)**

Functions: review

- f: A $\rightarrow$ C
- f: A x B $\rightarrow$ C

Examples:

- f: **N** $\rightarrow$ **N**, f(x) = 2x
- f: **N** x **N** $\rightarrow$ **N**, f(x,y) = x + y
- f: A x B $\rightarrow$ A, A = {a,b}, B = {0,1}

	0	1
a	a	b
b	b	a

Functions: an alternate view

Functions as lists of pairs or k-tuples

- E.g. f(x) = 2x
- {(1,2), (2,4), (3,6),….}
- For the function below:

  {(a,0,a),(a,1,b),(b,0,b),(b,1,a)}

	0	1
a	a	b
b	b	a

Next: Terminology

- Alphabets
- Strings
- Languages
- Problems, decision problems

Alphabets

- An alphabet is a finite non-empty set.
- An alphabet is generally denoted by the symbols Σ, Γ.
- Elements of Σ, called *symbols*, are often denoted by lowercase letters, e.g., a,b,x,y,..

Strings (or words)

- Defined over an alphabet Σ
- Is a finite sequence of symbols from Σ
- Length of string w (|w|) – length of sequence
- ε – the empty string is the unique string with zero length.
- Concatenation of w_1 and w_2 – copy of w_1 followed by copy of w_2
- $x^k = x\ x\ x\ x\ x \ldots x$(k times)
- w^R - reversed string; e.g. if w = abcd then w^R = dcba.
- Lexicographic ordering : definition

Languages

- A language over Σ is a set of strings over Σ
- Σ^* is the set of all strings over Σ
- A language L over Σ is a subset of Σ^* ($L \subseteq \Sigma^*$)
- Typical examples:

- $\Sigma=\{0,1\}$, the possible words over Σ are the finite bit strings.

- $L = \{ x \mid x$ is a bit string with two zeros $\}$

- $L = \{ a^n b^n \mid n \in \mathbf{N} \}$

- $L = \{1^n \mid n$ is prime$\}$

Concatenation of languages

Concatenation of two langauges:

$A \bullet B = \{ xy \mid x \in A \text{ and } y \in B \}$

Caveat: Do not confuse the concatenation of languages with the Cartesian product of sets.

For example, let $A = \{0,00\}$ then

$A \bullet A = \{ 00, 000, 0000 \}$ with $|A \bullet A|=3$,

$A \times A = \{ (0,0), (0,00), (00,0), (00,00) \}$
with $|A \times A|=4$

Problems and Languages

- Problem: defined using input and output
  - compute the shortest path in a graph
  - sorting a list of numbers
  - finding the mean of a set of numbers.
- Decision Problem: output is yes/no (or 1/0)
- Language: set of all inputs where output is yes

Historical perspective

- Many models of computation from different fields
  – Mathematical logic
  – Linguistics
  – Theory of Computation

Formal language theory

Input/output vs decision problems

Input/output problem: "find the mean of n integers"

Decision Problem: output is either yes or no

"Is the mean of the n numbers equal to k ?"

You can solve the decision problem if and only if you can solve the input/output problem.

Example – Code Reachability

- Code Reachability Problem:
  - Input: Java computer code
  - Output: Lines of unreachable code.
- Code Reachability Decision Problem:
  - Input: Java computer code and line number
  - Output: Yes, if the line is reachable for some input, no otherwise.
- Code Reachability Language:
  - Set of strings that denote Java code and reachable line.

Example – String Length

- Decision Problem:
  - Input: String w
  - Output: Yes, if |w| is even
- Language:
  - Set of all strings of even length.

Relationship to functions

- Use the set of k-tuples view of functions from before.
- A function is a set of k-tuples (words) and therefore a language.
- Shortest paths in graphs – the set of shortest paths is a set of paths (words) and therefore a language.

Recognizing languages

- Automata/Machines **accept** languages.
- Also called “recognizing languages”.

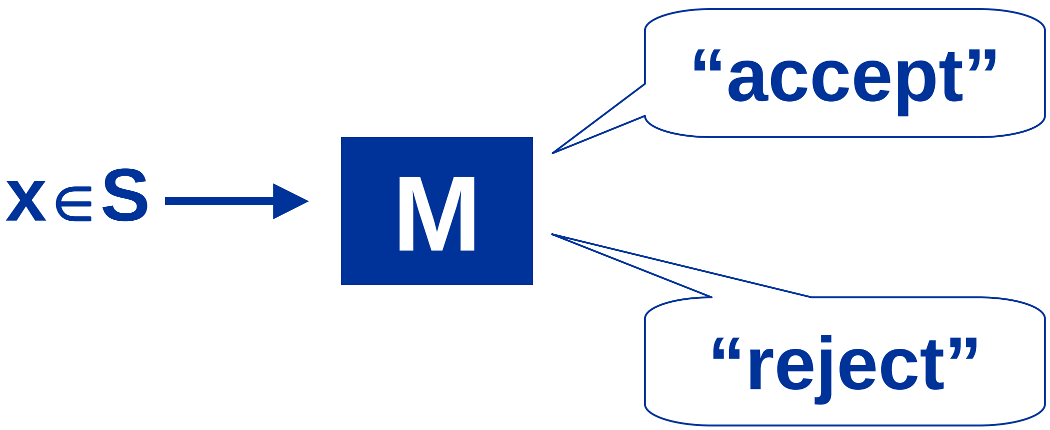
- The power of a computing model is related to, and described by, the languages it accepts/recognizes.

- Tool for studying different models

Recognizing Languages - 2

- Let L be a language $\subseteq$ S

- a machine M <u>recognizes</u> L if



“accept” if and only if $x \in L$

“reject” if and only if $x \notin L$

Recognizing languages - 3

- Minimal spanning tree problem solver

Recognizing languages - 4

- Tools from language theory
- Expressibility of languages
- Fascinating relationship between the complexity of problems and power of languages

Proofs

- What is a proof?
- Does a proof need mathematical symbols?
- What makes a proof incorrect?
- How does one come up with a proof?

Proof techniques (Sipser 0.4)

- Proof by cases.
- Proof by contrapositive
- Proof by contradiction
- Proof by construction
- Proof by induction
- Others …..

Proof by cases

If n is an integer, then $n(n+1)/2$ is an integer

- Case 1: n is even.

  or $n = 2a$, for some integer a

  So $n(n+1)/2 = 2a*(n+1)/2 = a*(n+1)$,

  which is an integer.

- Case 2: n is odd.

  $n+1$ is even, or $n+1 = 2a$, for an integer a

  So $n(n+1)/2 = n*2a/2 = n*a$,

  which is an integer.

Proof by contrapositive - 1

If x^2 is even, then x is even

- Proof 1 (DIRECT):

  $x^2 = x*x = 2a$

  So 2 divides x.

- Proof 2: prove the contrapositive!

  if x is odd, then x^2 is odd.

  $x = 2b + 1$. So $x^2 = 4b^2 + 4b + 1$ (odd)

Proof by contrapositive - 2

If $\sqrt{(pq)} \neq (p+q)/2$, then $p \neq q$

Proof 1: By squaring and transposing

$(p+q)^2 \neq 4pq$, or

$p^2+q^2 + 2pq \neq 4pq$, or

$p^2+q^2 - 2pq \neq 0$, or

$(p-q)^2 \neq 0$, or

$p-q \neq 0$, or $p \neq q$.

Proof 2: prove the contrapositive!

If $p = q$, then $\sqrt{(pq)} = (p+q)/2$

Easy: $\sqrt{(pq)} = \sqrt{(pp)} = \sqrt{(p^2)} = p = (p+p)/2 = (p+q)/2$.

Proof by contradiction

√2 is irrational

- Suppose √2 is rational. Then $\sqrt{2} = p/q$, such that p, q have no common factors.
  Squaring and transposing,
  $p^2 = 2q^2$ (even number)
  So, p is even (previous slide)
  Or $p = 2x$ for some integer x
  So $4x^2 = 2q^2$ or $q^2 = 2x^2$
  So, q is even (previous slide)
  So, p,q are both even – they have a common factor of 2. CONTRADICTION.
  So √2 is NOT rational. Q.E.D.

Proof by construction

There exists irrational b,c, such that b^c is rational

Consider $\sqrt{2}^{\sqrt{2}}$. Two cases are possible:

- Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational – DONE ($b = c = \sqrt{2}$).
- Case 2: $\sqrt{2}^{\sqrt{2}}$ is **irrational** – Let $b = \sqrt{2}^{\sqrt{2}}$, $c = \sqrt{2}$.

  Then $b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}*\sqrt{2}} = (\sqrt{2})^2 = 2$

Debug this "proof"

For each positive real number a, there exists a real number x such that $x^2 > a$

Proof: We know that $2a > a$

So $(2a)^2 = 4a^2 > a$

So use $x = 2a$.

Proof by induction

Format:

- Inductive hypothesis,
- Base case,
- Inductive step.

Proof by induction

Prove: For any $n \in \mathbf{N}$, n^3-n is divisible by 3.

IH: P(n): For any $n \in \mathbf{N}$, $f(n)=n^3-n$ is divisible by 3.
Base case: P(1) is true, because f(1)=0.
Inductive step:
Assume P(n) is true. Show P(n+1) is true.
Observe that $f(n+1) – f(n) = 3(n^2 + n)$
So f(n+1) – f (n) is divisible by 3.
Since P(n) is true, f(n) is divisible by 3.
So f(n+1) is divisible by 3.
Therefore, P(n+1) is true.
Exercise: give a direct proof.

Next: Finite automata

Ch. 1: Deterministic finite automata (DFA)

Look ahead:

We will study languages recognized by finite automata.