CSE 2001: Introduction to the theory of computation
Assignment 1 (Released Sept 11, 2012)
Submission deadline: 3:45 pm (in the dropbox) or 4 pm (in class), September 25, 2012
Solutions

Question 1: Sets

Is it possible for two distinct, nonempty sets A, B to satisfy $A \times B \subseteq B \times A$? Give either an example of sets A, B for which this is true or prove that this is not possible.
Solution: Since A, B are unequal, at least one of $B - A$, $A - B$ must be non-empty. We consider these cases separately.
Case 1: If $A - B$ is non-empty, let us assume that there is an element $a \in A - B$. Then consider any ordered pair (a, x) where $x \in B$. By the definition of cartesian product, $(a, x) \in A \times B$, but $(a, x) \notin B \times A$. Therefore $A - B \neq \phi$ implies that $A \times B \subseteq B \times A$ is false.
Case 2:
If $B - A$ is non-empty, let us assume that there is an element $b \in B - A$. Then any ordered pair (x, b) where $x \in A$ belongs to $A \times B$ but does not belong to $B \times A$.
Thus the given statement is false.

Question 2: Induction

Use strong induction to prove that every positive integer n can be written as a sum of distinct powers of 2. You will not get credit for this question if your proof is not based on strong induction.
[Hint: for the inductive step, separately consider the case where $k + 1$ is even and where it is odd.]
Solution:
Base case: 1 can only be represented one way: $1 = 1 * 2^0$.
Inductive step: As the hint suggests, let us assume that the statement is true for all $0 < m \leq k$. Now we consider 2 cases:
Case 1: If $k + 1$ is even, $k + 1 = 2m$ for some positive integer m. Since m can be written as a sum of distinct powers of 2 by the inductive hypothesis, so can 2m – multiplying by 2 simply increases the value of each exponent by 1.
Case 2: If $k + 1$ is odd, then k is even. By the inductive hypothesis, k can be written as a sum of distinct powers of 2. However, it cannot contain the term 2^0 since all other powers of 2 are even and the presence of 2^0 would make k odd. Therefore if we add 2^0 to the sum of powers of 2 that equals k, we get a new sum of distinct powers of 2 that equals $k + 1$.

Question 3

Give a direct proof, a proof by contraposition and a proof by contradiction of the statement: "If n is even, then $n + 4$ is even".
Solution:
Direct proof: If n is even, $n = 2k$ for some integer k. So $n + 4 = 2k + 4 = 2(k + 2)$ is even because $k + 2$ is an integer.
Proof by contraposition: If $n + 4$ is odd, then $n + 4 = 2k + 1$ for some integer k. So $n = 2k - 3 = 2(k - 2) + 1$. Since $k - 2$ is an integer, n must be odd.

Proof by contradiction: Suppose the statement does not hold for all n. So there exists some integer m for which m is even but $m+4$ is odd. Thus $m+3$ must be even. Since m is even, $m = 2k$ for some integer k. So $m+3 = 2k+3 = 2(k+1)+1$ must be odd, which is a contradiction.

Question 4

In each part below, a recursive definition is given of a subset of $\{a,b\}^*$. Give a simple nonrecursive definition in each case. Assume that each definition includes an implicit last statement: "Nothing is in L unless it can be obtained by the previous statements."

1. $a \in L$; for any $x \in L$, xa, xb are in L.

   Solution: L contains all strings starting with an a.

2. $a \in L$; for any $x \in L$, bx, xb are in L.

   Solution: L contains all strings of the form $b^m a b^n$, where $m, n \geq 0$.

3. $a \in L$; for any $x \in L$, ax, xb are in L.

   Solution: L contains all strings of the form $a^m a b^n$, where $m, n \geq 0$ or equivalently $a^m b^n$, $m > 0, n \geq 0$.

4. $a \in L$; for any $x \in L$, ax, bx, xb are in L.

   Solution: L contains all strings of the form Sab^n, where $n \geq 0$ and S is any string over $\{a, b\}$.

Question 5

Suppose a language L is defined recursively as:
$\epsilon \in L$; for every x, y in L, $axby$ and $bxay$ are both in L; nothing else is in L. Prove that L is precisely the set of strings in $\{a,b\}^*$ with equal numbers of a'a and b's.
Solution:
Let us first prove that all strings produced by the rules above have equal numbers of a, b's. Then by rule 3, all strings in L have equal numbers of a, b.
Most such proofs use induction on the length $|x|$ of strings produced – it is tailormade for recursively defined sets.
Base case: $|x| = 0$. The statement holds because ϵ contains an equal number (zero) of a, b's.
Inductive step: Assuming x, y satisfy the hypothesis, we see that both $axby, bxay$ have equal numbers of a, b's. Thus each string in L has an equal numbers of a, b's.
In order to prove the other direction, we need to show that any string with equal number of a, b's can be produced by the rules used to define L.
We can use induction again. For the base case, we use $|x| = 0$ again. We know this is in L. Inductively we assume that every string of length $2m$ or less with equal numbers of a, b's can be produced by the rules used to define L.
Now consider any such string s of length $2m+2$. Assume that s starts with an a. For each prefix of s, consider the number of a's minus the number of b's. This number is 1 after the prefix of length 1 and 0 at the end. It can change by 1 each time the prefix length is increased by 1, so it must hit 0 for the first time somewhere (at or before the end). Call this prefix p. We see that $p = axb$ for some string y. Let the suffix of p be y. Then the strings x, y have length less than or equal to $2m$. By the inductive hypothesis they are in L. Therefore s is produced by the rule $axby \in L$ and is in L.