MATH/CSE 1019 Test 3

1. (5 points) Use mathematical induction to prove that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$$

whenever n is a positive integer.

Proof:

Basis Step: n=1. L.H.S=$1^2 = 1$. R.H.S. = $\frac{1}{3} * 1 * (4 * 1^2 - 1) = 1$. L.H.S.=R.H.S. The statement is True.

Inductive Step: Assume the statement is true when n=k. Then

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{1}{3}k(4k^2-1).$$

Prove the statement is true when n=k+1, i.e.

$$1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2 = \frac{1}{3}(k+1)(4(k+1)^2-1)$$

L.H.S. = $1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2$

$=\frac{1}{3}k(4k^2-1) + (2(k+1)-1)^2$ <u>by the assumption</u>

$=\frac{1}{3}(4k^3-k) + (2k+1)^2$

$=\frac{1}{3}(4k^3-k) + (4k^2+4k+1)$

$=\frac{1}{3}(4k^3-k+12k^2+12k+3)$

$=\frac{1}{3}(4k^3+12k^2+11k+3)$

R.H.S. = $\frac{1}{3}(k+1)(4(k+1)^2-1)$

$=\frac{1}{3}(k+1)(4(k^2+2k+1) \quad -1)$

$=\frac{1}{3}(k+1)(4k^2+8k+3 \quad)$

$=\frac{1}{3}(4k^3+8k^2+3k+4k^2+8k+3 \quad)$

$=\frac{1}{3}(4k^3+12k^2+11k+3)$ =L.H.S.

By mathematical induction, the statement is true for all positive integers.

2. (5 points) Use strong induction to prove that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. (For example, 5 =2^0 + 2^2.Hint: For the inductive step, separately consider the case where k+1 is even and where it is odd.)

Proof:

<u>If n can be written as a sum of distinct powers of two, let us denote this sum as f(n).</u>

Basis Step: When n=1, $1 = 2^0$. The statement is true.

Inductive Step: Assume the statement is true when n=j, for $1 \leq \text{j} \leq \text{k}$. Then j can be written as a sum of distinct powers of two.

Prove the statement is true when n=k+1, i.e. (k+1) can be written as a sum of distinct powers of two.

(1) Case 1: k+1 is an odd positive integer.
Then k is an even positive integer.
By the assumption, k can be written as sum of distinct powers of two.
Since k is even, 2^0 is not included (otherwise k will be odd.)
Then k+1 can be written as f(k+1)= f(k)+ 2^0, which is sum of distinct powers of two.

(2) Case 2: k+1 is even.
Then (k+1)/2 is a positive integer, and $1 \leq (\text{k}+1)/2 \leq \text{k}$.
By the assumption, $(\text{k}+1)/2$ can be written as sum of distinct powers of two.
Then k+1 can be written as f(k+1) = 2*f($(\text{k}+1)/2$).
Since all the powers in f($(\text{k}+1)/2$) are distinct, then all the powers in 2*f($(\text{k}+1)/2$) are distinct too.

By strong induction, the statement is true.

3. (5 points) Use the definition of big-O and big-Ω to prove that $2n^3 + 7n^2 + 5$ is $\Theta(n^3)$.

Proof:
$2n^3 + 7n^2 + 5$ is $\Theta(n^3) \equiv$ $2n^3 + 7n^2 + 5$ is $O(n^3)$ and $2n^3 + 7n^2 + 5$ is $\Omega(n^3)$.
(1) Prove $2n^3 + 7n^2 + 5$ is $O(n^3)$.
We need to find a pair of witnesses C_1 and k_1, such that $\forall n > k_1$,
$|2n^3 + 7n^2 + 5| \leq C_1|n^3|$.
Notice that when $n > 1, n^3 > n^2 > 1$.
Then $|2n^3 + 7n^2 + 5| = 2n^3 + 7n^2 + 5 < 2n^3 + 7n^3 + 5n^3 = 14n^3$=$14|n^3|$

Let $C_1 = 14$ and $k_1 = 1$, then $\forall n > k_1$, $|2n^3 + 7n^2 + 5| \le C_1|n^3|$ is always true.

(2) Prove $2n^3 + 7n^2 + 5$ is $\Omega(n^3)$.

We need to find a pair of witnesses C_2 and k_2, such that $\forall n > k_2$, $|2n^3 + 7n^2 + 5| \ge C_2|n^3|$.

Notice that when $n > 0$, $7n^2 + 5 > 0$.

Then $|2n^3 + 7n^2 + 5| = 2n^3 + 7n^2 + 5 > 2n^3 = 2|n^3|$

Let $C_2 = 2$ and $k_2 = 0$, then $\forall n > k_2$, $|2n^3 + 7n^2 + 5| \ge C_2|n^3|$ is always true.

4. (5 points) Use the definition of big-O to prove $3n^2 - 10$ is not O(n).

Proof by contradiction:

Assume $3n^2 - 10$ is O(n).

Then $\exists C$ and k, such that $\forall n > k, |3n^2 - 10| \le C|n|$.

Notice when $n > \sqrt{10}$, $0 < 2n^2 < 3n^2 - 10$.

If $|3n^2 - 10| \le C|n|$, then $2n^2 < Cn$.

For any given C and k, let n= max{C/2, k, $\sqrt{10}$}+1, the statement $\forall n > k, |3n^2 - 10| \le C|n|$ will be False.

This is a contradiction.

Then the original statement "$3n^2 - 10$ is not O(n)" is true.

5. (5 points) What is the output of the following algorithm? What is the running time of this algorithm in big-O notation?

Input: $A = \begin{bmatrix} A[1,1] & A[1,2] & \dots & A[1,n] \\ A[2,1] & A[2,2] & \dots & A[2,n] \\ & \dots & & \\ A[n,1] & A[n,2] & \dots & A[n,n] \end{bmatrix}$

```
for i←1 to n
        for j←1 to n
                dotmp←A[i,j]
                A[i,j]←A[j,i]
                A[j,i]←tmp
return(A)
```

Solution:

Output: A

<u>Note: The elements will be swapped twice.</u>

Running Time: $O(n^2)$

6. (5 points) Using a loop invariant to prove that the following program for computing the factorials is correct. (Hint: n! = 1×2×3×…× (n-1)×n)

```
Input: A positive integer y
Output: y!

F(y)
1.      x←1
2.      z←1
3.      while x≤y
4.          do z←z*x
5.          x←x+1
6.      return(z)
```

Proof:

Loop invariant: Before the kth iteration, $x_k = k \leq (y+1)$, and z_k = (k-1)!.

<u>Note: Let us denote the values of x and z before the kth iteration as x_k and z_k.</u>

Proof:

Basis Case: k=1. Before the 1^{st} iteration, $x_1 = 1 \leq (y+1)$, and $z_1 = 0! = 1$. The loop invariant holds.

Inductive Step: Suppose the loop invariant holds before the kth iteration, i.e. before the

kth iteration, $x_k = k \leq (y+1)$, and z_k = (k-1)!.

If the loop does not terminate before this iteration, prove the loop invariant holds before the (k+1)th iteration, i.e. before the (k+1)th iteration, $x_{k+1} = (k+1) \leq (y+1)$, and z_{k+1} = k!.

In the 4^{th} line, $z_{k+1} = z_k * x_k = (k-1)! * k = k!$

In the 5^{th} line, $x_{k+1} = x_k + 1 = k+1$

Since the loop does not terminate before this iteration, then $x_k = k \leq y$, according to line 3. Therefore, $x_{k+1} = k+1 \leq y+1$.

∴ The loop invariant holds before the (k+1)th iteration.

Termination: x starts from 1, and keeps incrementing by 1 each time. It will be

eventually greater than y. So the algorithm will terminate, and k=y+1 upon termination. By the loop invariant, z_k = (k-1)!=y!.

Therefore, we have proved the correctness of the given algorithm.