CSE6338: Assignment 1

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1. Find the homogeneous transformation T_1^0 where:

   (a) $\{1\}$ has the same orientation as $\{0\}$ and the origin of $\{1\}$ is translated relative to the origin of $\{0\}$ by $d_1^0 = \begin{bmatrix}1 & -1 & 2\end{bmatrix}^T$.

   (b) The origin of $\{1\}$ is coincident with the origin of $\{0\}$, and $\hat{x}_1^0 = \hat{y}_0^0$, $\hat{y}_1^0 = -\hat{z}_0^0$, and $\hat{z}_1^0 = -\hat{x}_0^0$.

   (c) The origin of $\{0\}$ is translated relative to the origin of $\{1\}$ by $d_0^1 = \begin{bmatrix}0 & 0 & -1.7321\end{bmatrix}^T$, and $\hat{x}_1^0 = \begin{bmatrix}0.7887 & -0.2113 & -0.5774\end{bmatrix}^T$, $\hat{y}_1^0 = \begin{bmatrix}-0.2113 & 0.7887 & -0.5774\end{bmatrix}^T$, and $\hat{z}_1^0 = \begin{bmatrix}0.5774 & 0.5774 & 0.5774\end{bmatrix}^T$.

2. Consider the following 4×4 homogeneous transformation matrices:

$$\begin{aligned}
R_{x,a} &: \text{rotation about } x \text{ by an angle } a\\
R_{y,a} &: \text{rotation about } y \text{ by an angle } a\\
R_{z,a} &: \text{rotation about } z \text{ by an angle } a\\
D_{x,a} &: \text{translation along } x \text{ by a distance } a\\
D_{y,a} &: \text{translation along } y \text{ by a distance } a\\
D_{z,a} &: \text{translation along } z \text{ by a distance } a
\end{aligned}$$

   Write the matrix product giving the overall transformation for the following sequences (do not perform the actual matrix multiplications):

   (a) The following rotations all occur in the moving frame.

      i. Rotate about the current z-axis by angle ϕ.
      ii. Rotate about the current y-axis by angle θ.
      iii. Rotate about the current z-axis by angle ψ.

      Note: This yields the ZYZ-Euler angle rotation matrix.

   (b) The following rotations all occur in a fixed (world) frame.

      i. Rotate about the world x-axis by angle ψ.
      ii. Rotate about the world y-axis by angle θ.
      iii. Rotate about the world z-axis by angle ϕ.

      Note: This yields the roll, pitch, yaw (RPY) rotation matrix.

3. A rotation matrix in 3D is defined in terms of dot products. Prove the following statement: It does not matter what frame is used to compute the dot products as long as all of the dot products are computed using the same frame.

   In other words, prove that the dot product of two vectors $u \cdot v$ does not depend on the choice of coordinate frame. *Hint:* $u \cdot v = u^T v$

4. Consider a vector v that is rotated about a unit vector $\hat{k}$ (passing through the origin) by an angle θ to form a new vector v':

$$v' = R_{k,\theta} v$$

   Derive Rodrigues' rotation formula,

$$v' = v\cos\theta + (\hat{k} \times v)\sin\theta + \hat{k}(\hat{k} \cdot v)(1 - \cos\theta)$$

   Do not replicate the Wikipedia derivation; instead, use the rotation matrix for rotation about an axis $\hat{k}$ by an angle θ.

5. Treatment of bony deformity often requires cutting the deformed bone, followed by re-alignment of the bone fragments, followed by fixation. In computer-aided interventions, the re-alignment is often planned virtually using models derived from pre-operative medical images. To assess the accuracy of a performed procedure, it is necessary to compare the actual re-alignment achieved to the planned correction.

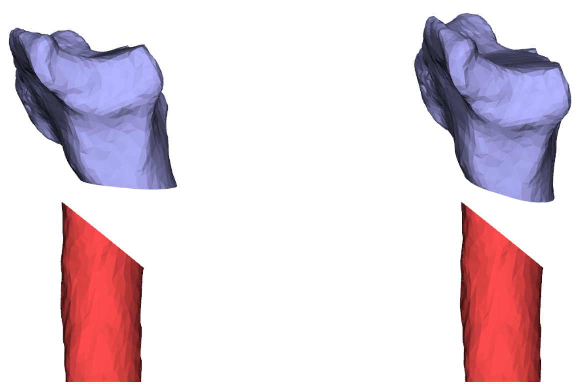

Figure 1: Left: The planned correction for a distal radial osteotomy. The transformation of the blue distal fragment relative to the red proximal fragment is given as $T^p_{d,\text{plan}}$. Right: The actual correction achieved. The transformation of the blue distal fragment relative to the red proximal fragment is measured to be $T^p_{d,\text{actual}}$.

   Find an expression for the correction error Δ in terms of $T^p_{d,\text{plan}}$ and $T^p_{d,\text{actual}}$. Note that Δ is a homogeneous transformation matrix.

   Hint: Suppose that $T^p_{d,\text{plan}} = T_{z,-5}$ (a rotation of $-5°$ about z) and that $T^p_{d,\text{actual}} = T_{z,10}$ (a rotation of $10°$ about z); then the correction error is $\Delta = T_{z,15}$.

6. Reconsider the situation in Question 5. The planned correction is usually expressed in the coordinate frame of the medical image; in the example given in Question 5 this frame would be the CT coordinate frame $\{\text{CT}\}$. The actual correction would be measured in some other coordinate frame. If we were performing a laboratory study using plastic bones then the actual correction might be measured by

using a 3D laser scanner to scan the physical plastic specimen; thus the actual correction might be expressed in the laser scanner frame $\{\text{LS}\}$.

Given $T^{\text{CT}}_{\text{LS}}$ and the actual correction T_{actual} expressed in $\{\text{LS}\}$, find an expression for the actual correction expressed in $\{\text{CT}\}$.

7. If you have never used Matlab, then work through Chapters 1 and 2 of the *Getting Starting Guide*:

   `http://www.mathworks.com/help/pdf_doc/matlab/getstart.pdf`

   and read the document *Working with Functions in Files*

   `http://www.mathworks.com/help/techdoc/matlab_prog/f7-41453.html`

   before attempting this question.

   Implement Horn's absolute orientation method as a Matlab function. The header of your function should be:

```
function T = horn(L, R)
% HORN Horn's absolute orientation method
%   HORN(L, R) solves the 3D absolute orientation problem for
%   the data sets L and R.  Returns the homogeneous transformation
%   matrix T that maps L to R. L and R are column vector matrices
%   with 3 rows and n columns.
```

   You can find the eigenvalues and eigenvectors of a matrix using Matlab's `eig` function. You can convert a quaternion to a rotation matrix using the following Matlab function:

```
function R = mq(q)
% MQ Rotation matrix from quaternion.
%   R = MQ(Q) returns the 3x3 rotation matrix defined by the
%   quaternion Q = [qw qx qy qz].

R = ones(3,3);

R(1,1) = q(1)*q(1) + q(2)*q(2) - q(3)*q(3) - q(4)*q(4);
R(1,2) = 2*(q(2)*q(3) - q(1)*q(4));
R(1,3) = 2*(q(2)*q(4) + q(1)*q(3));

R(2,1) = 2*(q(2)*q(3) + q(1)*q(4));
R(2,2) = q(1)*q(1) + q(3)*q(3) - q(2)*q(2) - q(4)*q(4);
R(2,3) = 2*(q(3)*q(4) - q(1)*q(2));

R(3,1) = 2*(q(2)*q(4) - q(1)*q(3));
R(3,2) = 2*(q(3)*q(4) + q(1)*q(2));
R(3,3) = q(1)*q(1) + q(4)*q(4) - q(2)*q(2) - q(3)*q(3);
```

   Submit your solution by emailing it to me.