Recursion and Logarithms

CSE 2011

Winter 2011

Recursion (3.5)

- In some problems, it may be natural to define the problem in terms of the problem itself.
- Recursion is useful for problems that can be represented by a simpler version of the same problem.
- Example: the factorial function

  6! = 6 * 5 * 4 * 3 * 2 * 1

  We could write:

  6! = 6 * 5!

Recursion (cont.)

- Recursion is one way to decompose a task into smaller subtasks. At least one of the subtasks is a smaller example of the same task.
- The smallest example of the same task has a non-recursive solution.
- Example: the factorial function

  `n! = n*(n-1)! and 1! = 1`

3

Example: Factorial Function

- In general, we can express the factorial function as follows:

  `n! = n*(n-1)!`

  Is this correct? Well… almost.
- The factorial function is only defined for *positive* integers. So we should be more precise:

```
f(n) = 1          if n = 1
     = n*f(n-1)   if n > 1
```

4

Factorial Function: Pseudo-code

```
int recFactorial(int n){
   if(n == 0)
      return 1;
   else
      return n * recFactorial(n-1);
}
```

recursion **means that a function calls itself.**

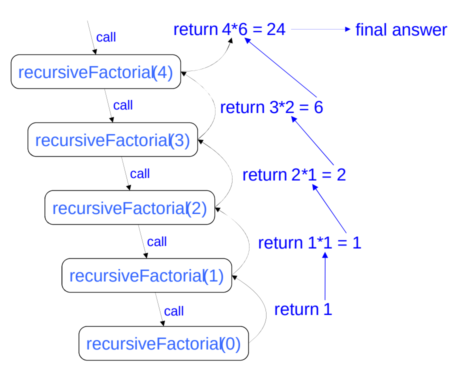
5

Visualizing Recursion

Recursion trace

- A box for each recursive call
- An arrow from each caller to callee
- An arrow from each callee to caller showing return value

Example recursion trace:



Using Recursion 6

Recursive vs. Iterative Solutions

- For certain problems (such as the factorial function), a recursive solution often leads to short and elegant code. Compare the recursive solution with the iterative solution:

```
int fac(int numb) {
  if (numb == 0)
    return 1;
  else
    return
      (numb*fac(numb-1));
}
```

```
int fac(int numb){
  int product = 1;
  while(numb > 1){
    product *= numb;
    numb--;
  }
  return product;
}
```

7

A Word of Caution

- To trace recursion, function calls operate as a stack – the new function is put on top of the caller.
- We have to pay a price for recursion:
  - calling a function consumes more time and memory than adjusting a loop counter.
  - high performance applications (graphic action games, simulations of nuclear explosions) hardly ever use recursion.
- In less demanding applications, recursion is an attractive alternative for iteration (for the right problems!)

8

Infinite Loops

If we use iteration, we must be careful not to create an infinite loop by accident.

```
for (int incr=1; incr!=10; incr+=2)
  ...

int result = 1;
while(result > 0){
  ...
  result++;
}
```

Oops!

Oops!

9

Infinite Recursion

Similarly, if we use recursion, we must be careful not to create an infinite chain of function calls.

```
int fac(int numb){
   return numb * fac(numb-1);
}

int fac(int numb){
   if (numb == 0)
      return 1;
   else
      return numb * fac(numb + 1);
}
```

Oops!
No termination condition

Oops!

10

Tips

We must always make sure that the recursion *bottoms out*:

- A recursive function must contain at least one non-recursive branch.
- The recursive calls must eventually lead to a non-recursive branch.

11

General Form of Recursion

- How to write recursively?

```
int recur_fn( parameters ){
  if ( stopping_condition )    // base case
      return stopping_value;
  if ( stopping_condition_2 )  // base case 2
      return stopping_value_2;
  return recur_fn( revised_parameters )
}
```

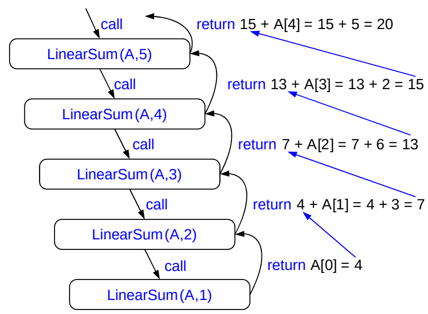
12

Example: Sum of an Array

Algorithm LinearSum(*A, n*):
Input:
A integer array *A* and an integer $n \geq 1$, such that *A* has at least *n* elements
Output:
Sum of the first *n* integers in *A*

if $n = 1$ **then**
 return *A*[0];
else
 return LinearSum(*A, n* - 1)
  + *A*[*n* - 1];

Example recursion trace:



Example: Reversing an Array

Algorithm ReverseArray(*A, i, j*):
Input: An array *A* and nonnegative integer indices *i* and *j*
Output: The reversal of the elements in *A* starting at index *i* and ending at *j*

if $i < j$ **then**
  Swap *A*[*i*] and *A*[*j*];
  ReverseArray(*A*, $i + 1$, $j - 1$);
return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(*A, i, j*), not ReverseArray(*A*).

Linear Recursion

- The above 2 examples use **linear recursion.**

Linear Recursion (2)

- **Test for base cases.**
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls **must** eventually reach a base case, and the handling of each base case should not use recursion.
- ***Recur once.***
  - Perform a single recursive call. (This recursive step may involve a test that decides which of several possible recursive calls to make, but it should ultimately choose to make just one of these calls each time we perform this step.)
  - Define each possible recursive call so that it makes progress towards a base case.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(*A, i, j*):
  Input: An array *A* and nonnegative integer indices *i* and *j*
  Output: The reversal of the elements in *A* starting at index *i* and ending at *j*
  while *i* < *j* **do**
    Swap *A*[*i*] and *A*[*j*]
    i = *i* + 1
    j = *j* - 1
  return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: binary search

Example: Binary Search

- Search for an element in a *sorted* array
  - Sequential search
  - Binary search
- Binary search
  - Compare the search element with the middle element of the array.
  - If not equal, then apply binary search to half of the array (if not empty) where the search element would be.

Binary Search with Recursion

```
// Searches an ordered array of integers using recursion
int bsearchr(const int data[], // input: array
             int first,        // input: lower bound
             int last,         // input: upper bound
             int value         // input: value to find
        ) // return index if found, otherwise return –1

{  int middle = (first + last) / 2;
   if (data[middle] == value)
      return middle;
   else if (first >= last)
      return -1;
   else if (value < data[middle])
      return bsearchr(data, first, middle-1, value);
   else
      return bsearchr(data, middle+1, last, value);
}
```

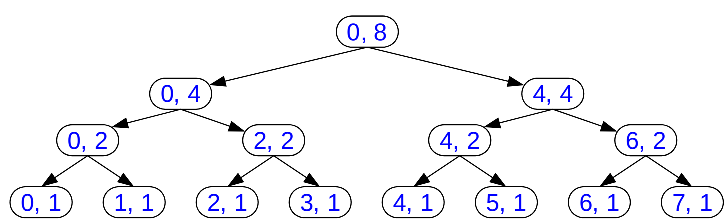
21

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(*A, i, n*):
  Input: An array *A* and integers *i* and *n*
  Output: The sum of the *n* integers in *A* starting at index *i*
 if *n* = 1 **then**
  return *A*[*i*];
 return BinarySum(*A, i, n/* 2) + BinarySum(*A, i* + *n/* 2, *n/* 2);

- **Example trace:** array A has 8 elements



22

Multiple Recursion

- Multiple recursion: makes potentially many recursive calls (not just one or two).
- Not covered in this course.

Running Time of Recursive Methods

- Could be just a hidden “for" or “while” loop.
  - See “Tail Recursion” slide.
  - “Unravel” the hidden loop to count the number of iterations.
- Logarithmic (next)
  - Examples: binary search, exponentiation, GCD
- Solving a recurrence
  - Example: merge sort (next lecture)

Logarithms

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Logarithmic Running Time

- An algorithm is O(logN) if it takes constant (O(1)) time to cut the problem size by a fraction (e.g., by ½).
- An algorithm is O(N) if constant time is required to merely reduce the problem by a constant amount (e.g., by 1).

Binary Search

```
int binarySearch (int[] a, int x)
{
/*1*/    int low = 0, high = a.size() - 1;
/*2*/    while (low <= high)
         {
/*3*/      int mid = (low + high) / 2;
/*4*/      if (a[mid] < x)
/*5*/         low = mid + 1;
/*6*/      else if (x < a[mid])
/*7*/         high = mid - 1;
           else
/*8*/         return mid;  // found
         }
/*9*/    return NOT_FOUND
}
```

27

Exponentiation x^n

```
long exp(long x, int n)
{
/*1*/    if (n==0)
/*2*/      return 1;
/*3*/    if (n==1)
/*4*/      return x;
/*5*/    if (isEven(n))
/*6*/      return exp(x*x, n/2);
         else
/*7*/      return exp(x*x, n/2)*x;
}
```

28

Euclid's Algorithm

- Computing the greatest common divisor (GCD) of two integers

```
long gcd (long m, long n) // assuming m>=n
{
/*1*/   while (n!=0)
         {
/*2*/     long rem = m%n;
/*3*/     m = n;
/*4*/     n = rem;
        }
/*5*/   return m;
}
```


Euclid's Algorithm (2)

- Theorem:
  - If M > N, then M mod N < M/2.
- Max number of iterations:
  - 2logN = O(logN)
- Average number of iterations:
  - $(12 \ln 2 \ln N)/\pi^2 + 1.47$

Next time …

- Merge Sort (section 11.1)
- Arrays, Linked Lists (chapter 3)

- Reading: section 3.5

31