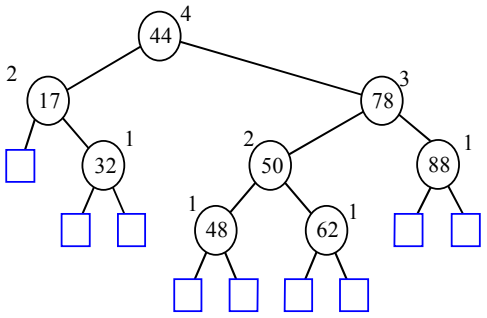
AVL Trees (10.2)

CSE 2011
Winter 2011

AVL Trees

- AVL trees are balanced.
- An AVL Tree is a ***binary search tree*** such that for every internal node v of T, the *heights of the children of v can differ by at most 1*.



An example of an AVL tree where the heights are shown next to the nodes

Height of an AVL Tree

- **Proposition**: The ***height*** of an AVL tree T storing n keys is O(log n).

Proof:

- Find **n(h):** the *minimum number of internal nodes* of an AVL tree of height h
- We see that n(1) = 1 and n(2) = 2
- For h ≥ 3, an AVL tree of height h contains the root node, one AVL subtree of height h−1 and the other AVL subtree of height h−2.
- i.e. n(h) = 1 + n(h−1) + n(h−2)

3

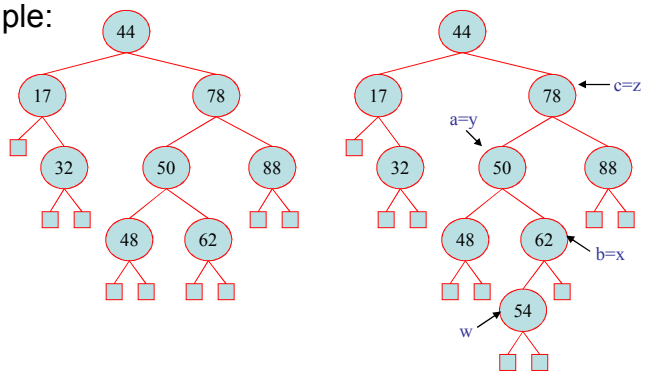
Height of an AVL Tree (2)

- Knowing n(h-1) > n(h-2), we get n(h) > 2n(h-2)
  n(h) > 2n(h-2)
  n(h) > 4n(h-4)
  …
  n(h) > 2^in(h-2i)
- Solving the base case we get: n(h) ≥ $2^{h/2-1}$
- Taking logarithms: h < 2log n(h) +2
- Thus the height of an AVL tree is O(log n)

4

Insertion in an AVL Tree

- Insertion is as in a binary search tree.
- Always done by expanding an external node.
- Example:



before insertion after insertion

5

Insertion: rebalancing

- A binary search tree T is called *balanced* if for every node *v*, the height of *v*'s children differ by at most 1.
- Inserting a node into an AVL tree involves performing *insertAtExternal*(*w, e*) on T, which changes the heights of some of the nodes in T.
- If an insertion causes T to become unbalanced, we travel up the tree from the newly created node w until we find the first node z that is unbalanced.
- y = child of z with higher height (Note: y = ancestor of w)
- x = child of y with higher height
  (Note: x = ancestor of w or x = w)
- Since z became unbalanced by an insertion in the subtree rooted at its child y, height(y) = height(sibling(y)) + 2

6

Insertion: restructuring

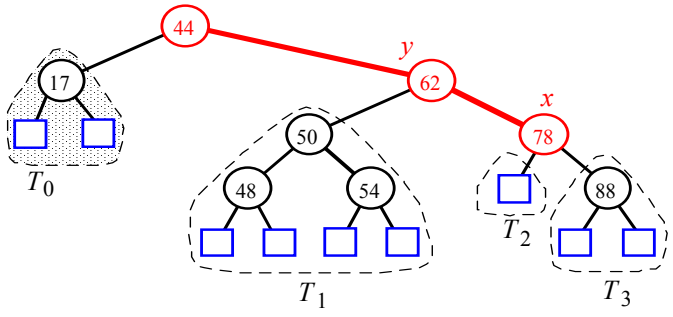
- Now to rebalance...
- To rebalance the subtree rooted at *z*, we must perform a *restructuring*.
- Two methods:

1. cut/link restructuring (not in textbook)
   - Given 7 integer keys 1, 2, 3, 4, 5, 6, and 7, how do we build a balanced BST?
2. tri-node restructuring (textbook)

7

Cut/Link Restructure Algorithm

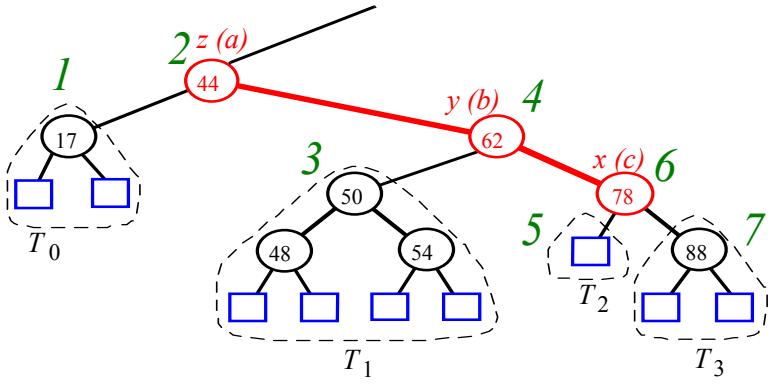
- Any tree that needs to be balanced can be grouped into 7 parts: x, y, z, and the 4 subtrees anchored at the children of those nodes (T_0, T_1, T_2, T_3).
- Any of the 4 subtrees can be empty (one dummy leaf).



8

Cut/Link Restructure Algorithm

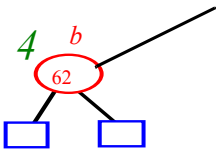
- Number the 7 parts by doing an inorder traversal.
- x,y, and z are now renamed based upon their order within the inorder traversal.



9

Cut/Link Restructure Algorithm

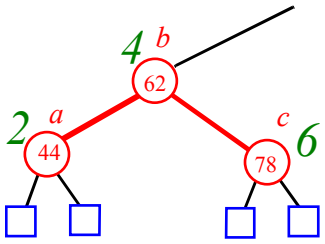
- Now we can re-link these subtrees to the main tree.
- Link in node 4 (b) where the subtree's root was.



10

Cut/Link Restructure Algorithm

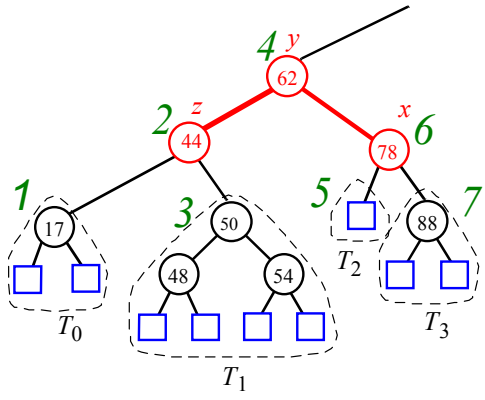
- Link in nodes 2 (a) and 6 (c) as children of node 4.



11

Cut/Link Restructure Algorithm

- Finally, link in subtrees 1 and 3 as the children of node 2, and subtrees 5 and 7 as the children of 6.



12

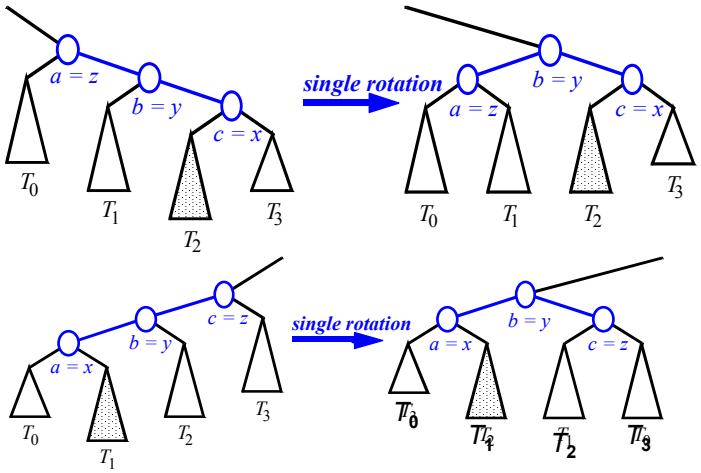
Tri-node Restructuring

- To rebalance the subtree rooted at z, we must perform a *restructuring*.
- We rename x, y, and z to a, b, and c based on the order of the nodes in an in-order traversal (see the next slides for 4 possible mappings).
- z is replaced by b, whose children are now a and c whose children, in turn, consist of the 4 other subtrees formerly children of x, y, and z.

13

Tri-node Restructuring (2)

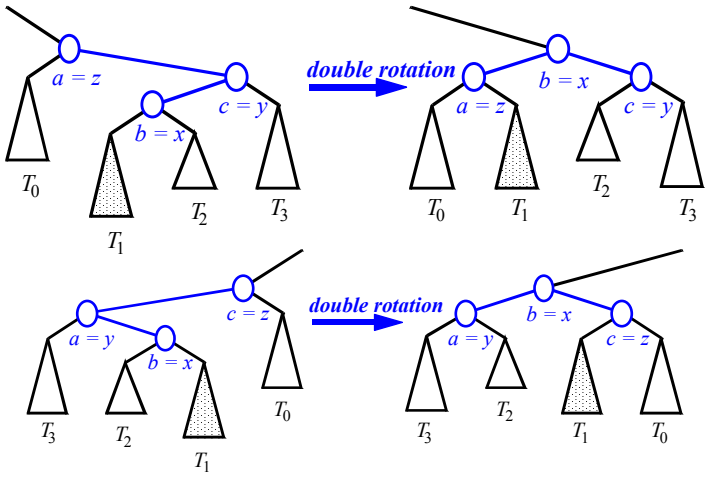
Single rotations



14

Tri-node Restructuring (3)

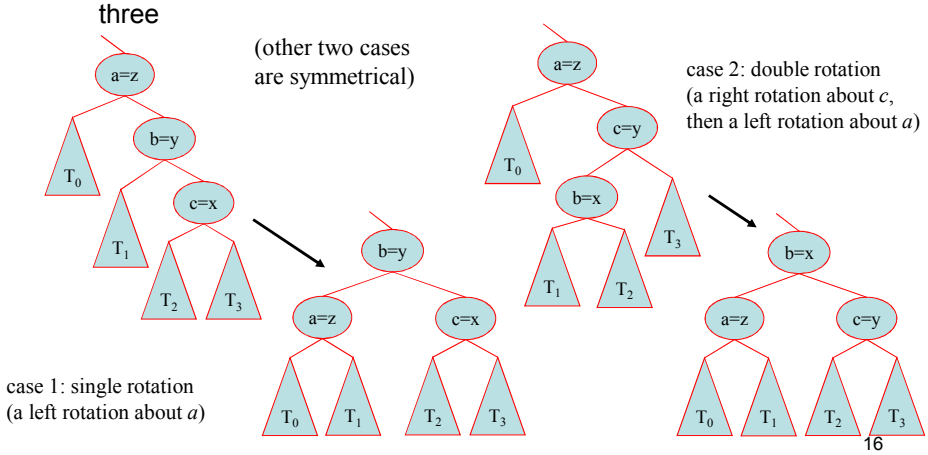
Double rotations



15

Single/Double Rotations

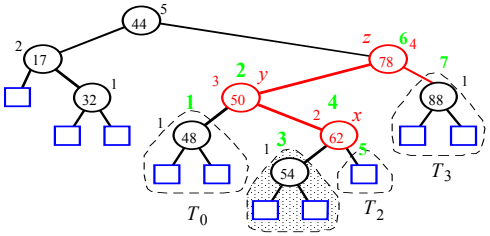
- let (*a*,*b*,*c*) be an inorder listing of *x*, *y*, *z*
- perform the rotations needed to make *b* the topmost node of the three



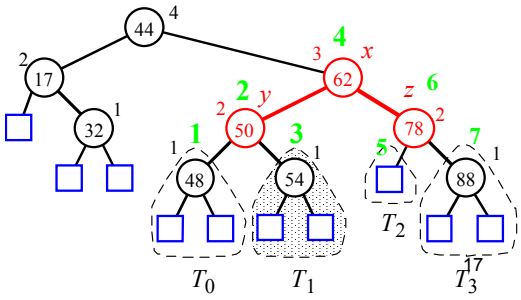
16

Restructuring Example

unbalanced...



...balanced



Restructure Algorithm

Algorithm restructure(x):

Input: A node *x* of a binary search tree T that has both a parent *y* and a grandparent *z*

Output: Tree T restructured by a rotation (either single or double) involving nodes x, y, and z.

1. Let (*a, b, c*) be an inorder listing of the nodes *x, y, and z,* and let (T0, T1, T2, T3) be an inorder listing of the the four subtrees of *x, y, and z,* not rooted at *x, y, or z.*
2. Replace the subtree rooted at z with a new subtree rooted at *b*
3. Let *a* be the left child of *b* and let T0, T1 be the left and right subtrees of *a*, respectively.
4. Let *c* be the right child of *b* and let T2, T3 be the left and right subtrees of *c*, respectively.

18

Cut/Link vs. Tri-node Restructuring

- Both methods give the same results.
- Both methods have the same time complexity.
- Time complexity = ?
- Cut/link method:
  - Advantage: no case analysis; more elegant.
  - Disadvantage: can be more code to write.

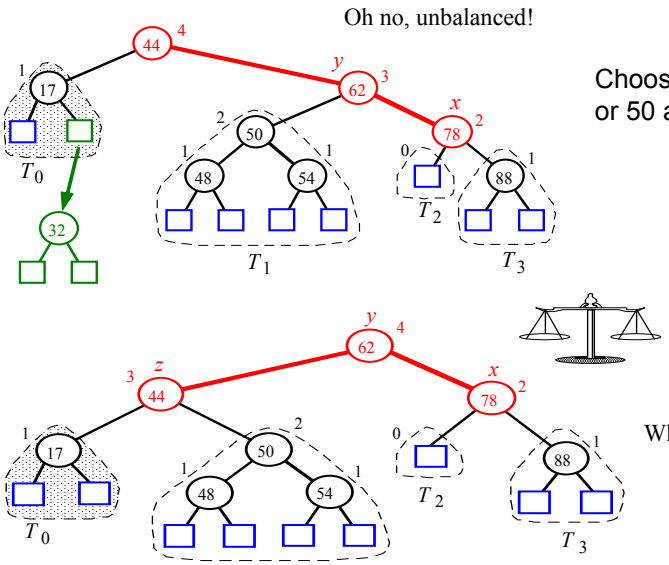
19

Removal

- First remove the node as in a BST.
- Performing a removeExternal(w) can cause T to become unbalanced.
- Let *z* be the first unbalanced node encountered while travelling up the tree from w.
- y = child of z with higher height (y ≠ ancestor of w)
- x = child of y with higher height, or either child if two children of y have the same height.
- Perform operation restructure(x) to restore balance at the subtree rooted at z.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached.

20

Removal Example

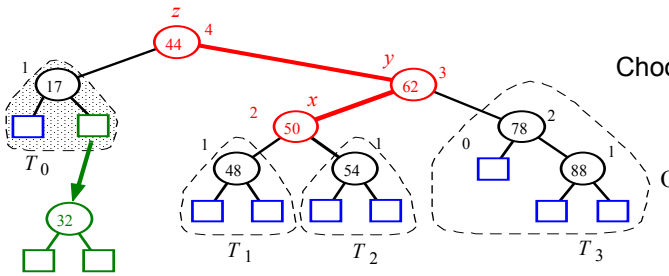


Choose either 78 or 50 as node *x*.

Whew, balanced!

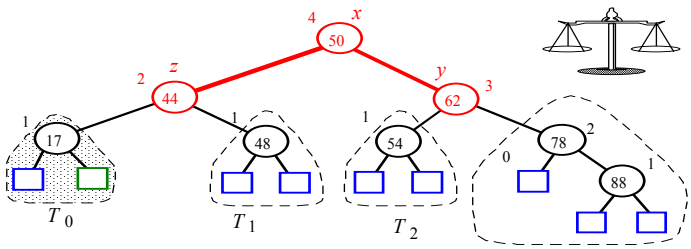
21

Removal Example (2)



Choose 50 as *x*.

Oh no, unbalanced!



Whew, balanced!

22

Next time …

- Heaps (8.3)

23