Homework Assignment #1
Due: January 26, 2011 at 2:30 p.m.

1. People who live in the Kingdom of Leutonia are notoriously superstitious. When they heard from other countries in the 12th century about the discovery of the number 0, the Leutonian high priest declared that 0 is Kurat's number[1], and declared that anyone who uses the number 0 would be excommunicated from Leutonian society. This limited the growth of mathematics and science in Leutonia for many centuries.

The first great Leutonian mathematician, Lisa Paio (1824–1907), devised a new Leutonian Numbering System (LNS) that avoided the use of 0. It uses two characters, 1 and 2, to represent positive integers as follows. If $s = s_{\ell-1}s_{\ell-2}\ldots s_2 s_1 s_0$ is a string of length ℓ, where each $s_i \in \{1, 2\}$, then the number represented by s is

$$n(s) = \sum_{i=0}^{\ell-1} s_i \cdot 2^i.$$

For example, 11221 represents the number

$$n(11221) = 1 \cdot 16 + 1 \cdot 8 + 2 \cdot 4 + 2 \cdot 2 + 1 = 37.$$

In this question, we want to verify that LNS really works: that is, that it represents each positive integer in a unique way.

(a) Write out the strings that represent the first 12 positive integers in LNS.

(b) Let $a(\ell)$ be the smallest integer that can be represented in LNS by a string of length ℓ. Give a simple expression for $a(\ell)$. You need not prove your answer is correct.

(c) Let $b(\ell)$ be the largest integer that can be represented in LNS by a string of length ℓ. Give a simple expression for $b(\ell)$. You need not prove your answer is correct.

(d) Prove the following claim: For every $\ell \geq 1$, every number in $\{x \in \mathbb{N} : a(\ell) \leq x \leq b(\ell)\}$ can be represented in LNS.

(e) Prove that every positive integer has a *unique* representation in LNS.

(f) Notice that the LNS representation has an advantage over binary representation of numbers. You showed above that every positive integer is represented by exactly one string in LNS. Binary representation, on the other hand can have multiple strings representing a single number if leading 0's are allowed. (For example, 000101 and 101 both represent 5 in binary.) Can you think of a *disadvantage* of LNS, as compared to binary representation?

[1] Kurat is a devil-like figure in Leutonian mythology