Prolog and the Resolution Method

The Logical Basis of Prolog

Chapter 10

Background

- Prolog is based on the **resolution proof** method developed by Robinson in 1966.
- **Complete** proof system with only one rule.
  - **If something can be proven from a set of logical formulae, the method finds it.**
- Correct
  - **Only theorems will be proven, nothing else.**
- Proof by contradiction
  - **Add negation of a purported theorem to a body of axioms and previous proven theorems**
  - **Show resulting system is contradictory**

Propositional Logic

- Infinite list of propositional variables
  - **a, b, … , z, p_1 … p_n , q_n … q_r, …**
- Logical connectives
  - **¬ ∧ ∨ → ↔**
- The set of fomula's of propositional logic is the smallest set, FOR, such that
  - **Every popositional variable is in FOR**
  - **If A and B are elements of FOR then ¬A, A ∧ B, A ∨ B, A → B, A ↔ B are elements of FOR**
- Every variable represents 0 or 1

Propositional clauses – informal

- ◊ Have a collection of clauses in conjunctive normal form
  - » **Each clause is a set of propositions connected with or**
  - » **Propositions can be negated (use not ~)**
  - » **set of clauses implicitly anded together**

- ◊ Example

A or B

C or D or ~ E

F

==>

(A or B) and (C or D or ~ E) and F

Clausal Form

◊ A clause is an expression of the following form, called **clausal form**

$$\underbrace{l_0, l_1, l_2, \ldots l_k}_{\text{commas are disjunctions}} \leftarrow \underbrace{d_0, d_1, d_2, \ldots d_m}_{\text{commas are conjunctions}}$$

$$a \leftarrow b \equiv a \lor \neg b$$

As a consequence the clausal form can be written as

$$l_0 \lor l_1 \lor l_2 \lor \ldots \lor l_k \lor \neg(d_0 \land d_1 \land d_2 \land \ldots \land d_m)$$

Using de'Morgans law

$$l_0 \lor l_1 \lor l_2 \lor \ldots \lor l_k \lor \neg d_0 \lor \neg d_1 \lor \neg d_2 \lor \ldots \lor \neg d_m$$

Conjunctive Normal Form

- If $S = \{c_0, c_1, c_2, \ldots c_k\}$ are a set of clauses then the representation of S is the formula

  $\alpha = \alpha_{c0} \wedge \alpha_{c1} \wedge \alpha_{c2} \wedge \ldots \wedge \alpha_{ck})$

- α is in CNF (conjunctive normal form)
- α_{ci} is a disjunction of variables and their negations
- α is a conjunction of these disjunctions

Every formula can be converted to CNF

Contradiction in a set of clauses

- The set $\{ p \wedge \neg p \}$ is a contradiction of clauses
- In clausal form this is

$$p \leftarrow$$
$$\leftarrow p$$

- We say that resolving upon p gives [ ] the empty clause which is false.

Propositional case – Resolution

- What happens if there is a contradiction in the set of clauses

- Example – only one clause

  P

- Add ~P to the set of clauses

  P
  ~ P
  ==>
  P and ~ P
  ==>
  [ ]   -- null the empty clause is false

- Think of **P** and **~P** canceling each other out of existence

Resolution rule

◊ Given the clause

Q or ~R

◊ and the clause

R or P

◊ then resolving the two clauses is the following

(Q or ~R) and (R or P)

==>

P or Q **-- new clause that can be added to the set**

◊ Combining two clauses with a positive proposition and its negation (called **literals**) leads to adding a new clause to the set of clauses consisting of all the literals in both parent clauses except for the literals resolved on

Resolution rule – 2

◊ Given the clause

L_1 or L_2 or … or L_p or ~R

◊ and the clause

R or K_1 or K_2 or … or K_q

◊ then resolving the two clauses is the following

(L_1 or L_2 or … or L_p or ~R) and (R or K_1 or K_2 or … or K_q)

==>

(L_1 or L_2 or … or L_p or K_1 or K_2 or … or K_q)

-- new clause that can be added to the set

Resolution method

◊ Combine clauses using resolution to find the empty clause

» **Implying one or more of the clauses in the set is false.**

◊ Given the clauses

1 P
2 ~P or Q
3 ~ Q or ~R
4 R

◊ Can resolve as follows

5	**P and (~P or Q) ==> Q**	**resolve 1 and 2**
6	**Q and (~Q or ~R) ==> ~R**	**resolve 5 and 3**
7	**~R and R ==> []**	**resolve 6 and 4**

Resolution method – 2

- ◊ Using resolution to prove a theorem
  - > 1 Given the non contradictory clauses – assuming original set of clauses is true

P
~P or Q
~ Q or ~R

  - > 2 Add the negation of the theorem, ~R , to be proven true

R

  - – Clause set now contains a contradiction
  - > 3 Find [] – showing that a contradiction exists, (see previous slide)
  - > 4 implies R is false, hence the theorem, ~ R, is true

Resolution method – 3

- In general resolution leads to longer and longer clauses
  - **Length 2 & length 2 --> length 2 (see earlier slide) – no shorter**
  - **Length 3 & length 2 –> length 3 (longer)**
  - **In general length p & length q --> length p + q - 2 (see earlier slide)**
- Non trivial to find the sequence of resolution rule applications needed to find []
- But at least there is only one rule to consider, which has helped automated theorem proving

The Big Question

How does all this relate to Prolog ?

If A then B – Propositional case

- Example 1: In prolog we write

  A :- B.

- Which in logic is

  A if B ==> if B then A

  ==> A or ~B

Clausal form
A ← B

- Example 2

  A :- B , C , D.

  A if B and C and D

  ==> if B and C and D then A

  ==> A or ~B or ~C or ~D

Clausal form
A ← B, C, D

If A then B – Propositional case – 2

◊ Example 2

if B and C and D then P and Q and R

==> ~B or ~C or ~D or (P and Q and R)

==> (~B or ~C or ~D) or (P and Q and R)

distribution

==> ~B or ~C or ~D or P
~B or ~C or ~D or Q
~B or ~C or ~D or R

Clausal form
P ← B, C, D
Q ← B, C, D
R ← B, C, D

> **In Prolog**

P :- B , C , D.
Q :- B , C , D.
R :- B , C , D.

If A then B – Propositional case – 4

- Example 3

  if B and C and D then P or Q or R

  ==> ~B or ~C or ~D or P or Q or R

  > **No single statement in Prolog for such an if ... then ..., choose one or more of the following depending upon the expected queries and database**

```
P  :-  B , C , D , ~Q , ~R
Q  :-  B , C , D , ~P , ~R
R  :-  B , C , D , ~P , ~Q
```

Clausal form
P, Q, R ← B, C, D

If A then B – Propositional case – 5

- ◊ Example 4

  if the_moon_is_made_of_green_cheese

  then pigs_can_fly

  ==>

  ~ the_moon_is_made_of_green_cheese or pigs_can_fly

  > **In Prolog**

  pigs_can_fly :-
  the_moon_is_made_of_green_cheese

Prolog facts – propositional case

◊ Prolog facts are just themselves.

A
P
the_moon_is_made_of_green_cheese
pigs_can_fly

◊ Comes from

if true then pigs_can_fly

==> pigs_can_fly or ~true
==> pigs_can_fly or false
==> pigs_can_fly

◊ In Prolog

pigs_can_fly :- true **:- true is implied,**
so it is not written

Query

◊ A query "**A and B and C**", when negated is equivalent to

if A and B and C then false

> **insert the negation into the database, expecting to find a contradiction**

◊ Translates to

false or ~A or ~B or ~C

==> ~A or ~B or ~C

Is it true pigs_fly?

- Add the negated question to the database

  If pigs_fly then false

  ==> ~pigs_fly or false ==> ~pigs_fly

- If the database contains

  pigs_fly

- Then resolution obtains **[]**, the contradiction, so the negated query is false, so the query is true.

- Prolog distinguishes between facts and queries depending upon the mode in which it is being used. In **(re)consult** mode we are entering facts. Otherwise we are entering queries.

A longer example

1 pigs_fly :- pigs_exist , animals_can_fly.

==> pigs_fly ∨ ~pigs_exist ∨ ~animals_can_fly

2 pigs_are_pink.

==> pigs_are_pink

3 pigs_exist.

==> pigs_exist

4 birds_can_fly.

==> birds_can_fly

5 animals_can_fly.

==> animals_can_fly

Hypothesize that pigs can fly

6 :- pigs_fly.

==> ~pigs_fly

A longer example – 2

Resolve 6 & 1 ==>
7 ~pigs_exist ∨ ~animals_can_fly

Resolve 7 & 3 ==>
8 ~animals_can_fly

Resolve 8 & 5 ==>
9 [ ]

We have the empty clause – a refutation
As a consequence, the negated statement is false,
the original statement is true.

Predicate Calculus

- Step up to predicate calculus as resolution is not interesting at the propositional level.
- We add
  - **the universal quantifier – for all x – $\forall$ x**
  - **the existential quantifier – there exists an x – $\exists$ x**
- But in Prolog there are no quantifiers?
  - **They are represented in a different way**

Forall x – ∀ x

- The universal quantifier is used in expressions such as the following

  $\forall\, x \cdot P (x)$

  > **For all x it is the case that P(x) is true**

  $\forall\, x \cdot lovesBarney (x)$

  > **For all x it is the case that lovesBarney(x) is true**

- The use of variables in Prolog takes the place of universal quantification – a variable implies universal quantification

  P (X)

  > **For all X it is the case that P(X) is true**

  lovesBarney (X)

  > **For all x it is the case that lovesBarney(X) is true**

Exists x – ∃ x

◊ The existential quantifier is used in expressions such as the following

$\exists x \cdot P(x)$

> There exists an x such that P(x) is true

$\exists x \cdot lovesBarney (x)$

> There exists an x such that lovesBarney(x) is true

◊ Constants in Prolog take the place of existential quantification – a constant implies existential quantification

– **The constant is a value of x that satisfies existence**

P (a)	a is an instance such that P(a) is true
lovesBarney (elliot)	elliot is an instance such that lovesBarney (elliot) is true

Nested quantification

◊ $\exists x \exists y \cdot sisterOf(x, y)$

> There exists an x such that there exists a y such that x is the sister of y

> In Prolog introduce two constants

sister (mary , eliza)

◊ $\exists x \forall y \cdot sisterOf(x, y)$

> There exists an x such that forall y it is the case that x is the sister of y

sister (leila , Y)

> One constant for all values of Y

Nested quantification – 2

◊ $\forall x \exists y \cdot \text{sisterOf}(x, y)$

> Forall x there exists a y such that x is the sister of y

> The value of y depends upon which X is chosen, so Y becomes a function of X

sisterOf (X , f (X))

◊ $\forall x \forall y \cdot \text{sisterOf}(x, y)$

> Forall x and forall y it is the case that x is the sister of y

sisterOf (X , Y)

> Two independent variables

Nested quantification – 3

◊ $\forall x \forall y \exists z \cdot P(z)$

> Forall x and for all y there exists a z such that P(z) is true

> The value of z depends upon both x and y, and so becomes a function of X and Y

$P(g(X, Y))$

◊ $\forall x \exists y \forall z \exists w \cdot P(x, y, z, w)$

> Forall x there exists a y such that forall z there exists a w such that P(x,y,z,w) is true

> The value of y depends upon x, while the value of w depends upon both x and z

$P(X, h(X), Z, g(X, Z))$

Skolemization

- ◊ Removing quantifiers by introducing variables and constants is called **skolemization**
- ◊ Removal of ∃ gives us functions and constants – functions with no arguments.
  - » **Functions in Prolog are called structures or compound terms**
- ◊ Removal of ∀ gives us variables
- ◊ Each predicate is called a **literal**

Herbrand universe

- The transitive closure of the constants and functions is called the **Herbrand universe** – in general it is infinite
- A Prolog database defines predicates over the Herbrand universe determined by the database

Herbrand universe – Determination

◊ It is the union of all constants and the recursive application of functions to constants

  » **Level 0 – Base level – is the set of constants**

  » **Level 1 constants are obtained by the substitution of level 0 constants for all the variables in the functions in all possible patterns**

  » **Level 2 constants are obtained by the substitution of level 0 and level 1 constants for all the variables in the functions in all possible patterns**

  » **Level n constants are obtained by the substitution of all level 0..n-1 constants for all variables in the functions in all possible patterns**

Back to Resolution

- Predicate calculus case is similar to the propositional case in that resolution combines two clauses where two literals cancel each other
- With variables and constants we use pattern matching to find the **most general unifier** (binding list for variables) between two literals
- The unifier is applied to all the literals in the two clauses being resolved
- All the literals, except for the two which were unified, in both clauses are combined with “or”
- The new clause is added to the set of clauses
- When [] is found, the bindings in the path back to the query give the answer to the query

Example

◊ Given the following clauses in the database

person (bob).

~person (X) or mortal (X).

forall X • if person (X) then mortal (X)

◊ Lets make a query asking if bob is a person

◊ The query adds the following to the database

~person (bob).

◊ Resolution with the first clause is immediate with no unification required

◊ The empty clause is obtained
So ~person(bob) is false, therefore person(bob) is true

Example – 2

- ◊ Given the following clauses in the database

  person (bob).

  ~person (X) or mortal (X).

  forall X • if person (X) then mortal (X)

- ◊ Lets make a query asking if bob is mortal

- ◊ The query adds the following to the database

  ~mortal (bob).

- ◊ Resolution with the second clause gives with **X_1 = bob** (renaming is required!)

  ~person (bob).

- ◊ Resolution with the first clause gives []
  So ~mortal(bob) is false, therefore mortal(bob) is true

Example – 3

◊ Given the following clauses in the database

person (bob).
~person (X) or mortal (X).

◊ Lets make a query asking does a mortal exist
The query adds the following to the database

~mortal (X). **~ (∃ x • mortal (x)) -- negated query**

◊ Resolution with the second clause gives with **X_1 = X**
(renaming is required!)

~person (X_1).

◊ Resolution with the first clause gives [] with **X_1 = bob**
So ~mortal(X) is false, therefore mortal(X) is true with
X = bob

Example – 4

- Given the following clauses in the database

  person (bob).
  ~person (X) or mortal (X).

- Lets make a query asking is alice mortal

  ~mortal (alice).

- Resolution fails with the first clause but succeeds with the second clause gives with **X_1 = alice**

  ~person (alice).

- Resolution with the first clause and second clause fails, searching the database is exhausted without finding []

- So ~mortal(alice) is true, therefore mortal(alice) is false

Example – 4 cont'd

- Actually all that the previous query determined is that ~mortal(alice) is consistent with the database and resolution was unable to obtain a contradiction

Prolog searches are based on a **closed universe**

Truth is relative to the database

Unification

- In order to use the resolution method with predicate calculus we need to be able to find the most general unifier (mgu) between two literals.
- **p(a, b, c)** and **p(X, Y, Z)**
  - » **mgu = { X / a , Y / b , Z / c }**
- **f(g(a, b), a, g(a, b)** and **f(g(X, Y, X, g(X, y))**
  - » **mgu = { X / a , Y / b , Z / a }**
- **p(a, f(b, a), c)** and **p(X, f(X, Y), Z)**
  - » **mgu does not exist**
- **p(X, a, b)** **and** **p(Y, Y, b)**
  - » **mgu = { X / Y , Y / a}**

Factoring

- ◊ General resolution permits unifying several literals at once by **factoring**
  - > **unifying two literals within the same clause - if they are of the same "sign", both positive, P(...) or P(...), or both negative, ~P(...) or ~P(...)**
- ◊ Why factor?
  - > **Gives shorter clauses, making it easier to find the empty clause**

Factoring – 2

◊ For example given the following clause

loves (X , bob) or loves (mary , Y)

◊ We can factor (obtain the common instances) by unifying the two loves literals

loves (mary , bob) **X = mary and Y = bob**

◊ The factored clause is implied by the un-factored clause as it represents a subset of the cases that make the un-factored clause true

> **Can be added to the database without contradiction**

Creating a database

- A large part of the work in creating a database is to convert general predicate calculus statements into conjunctive normal form.
- Much of Chapter 10 of Clocksin & Mellish describes how this can be done.

Horn clauses

◊ Clauses where the consequent is a single literal.

> **For example, X is the consequent in**

If A and B and C then X

◊ Horn clauses are important because, while resolution is complete, it usually leads to getting longer and longer clauses while finding contradiction means getting the empty clause

» **Need to get shorter clauses or at least contain the growth in clauses**

» **General resolution can lead to exponential growth in both**

> **clause size**

> **size of the set of clauses**

Horn clauses – 2

- Horn clauses have the property
  - > **Every clause has at most one positive literal (un-negated) and zero or more negative literals**

  person (bob).
  mortal (X) ~person (X)
  binTree (t (D , L , R))
  ~treeData (D) ~binTree (L) ~binTree (R).

- Facts are clauses with one positive literal and no negated literals, resolving with facts reduces the length of clauses

- Horn clauses can represent anything we can compute
  - » **Any database and theorem that can be proven within first order predicate calculus can be translated into Horn clauses**