Equivalence Relations

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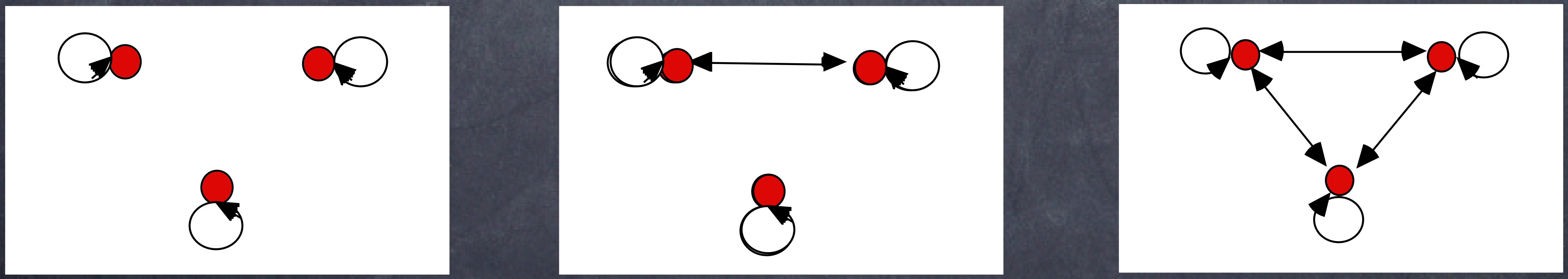
Overview

- Properties of relations on a set: reflexive, symmetric, and transitive
- Now we group these properties together to define new types of important relations

Equivalence Relations

- A relation R on a set A is an equivalence relation iff R is
  - reflexive
  - symmetric
  - and transitive.



Equivalence Relations Examples

- Let R be a relation on Z such that aRb iff a=b (or a-b=0).
- Let R be a relation on the set of real numbers such that aRb iff a-b is an integer.
- Let R be a relation on the set of bit strings of length three or more, such that aRb iff a and b agree in their first three bits

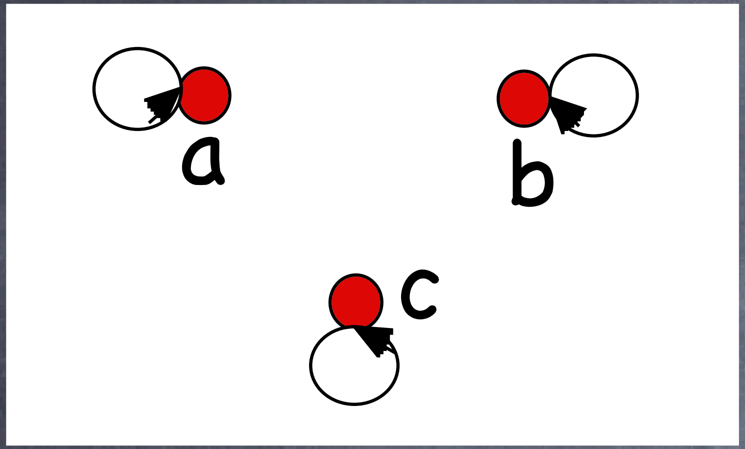
Non-equivalence Relations Examples

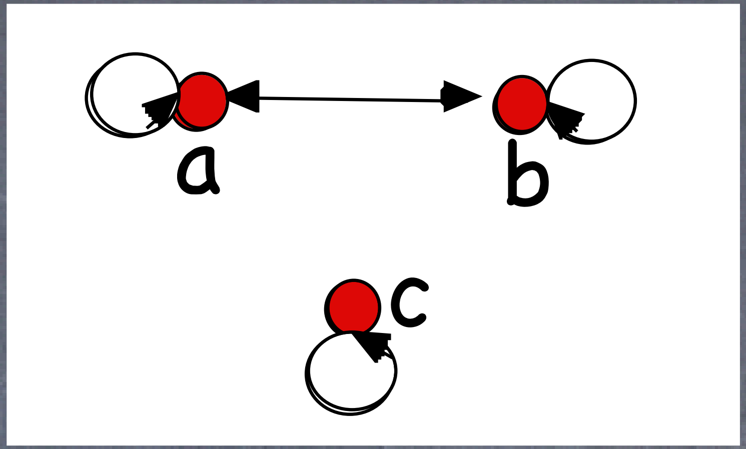
- The "divides" relation on Z^+
- Let R be a relation on the set of real numbers such that aRb iff $|a-b|<1$

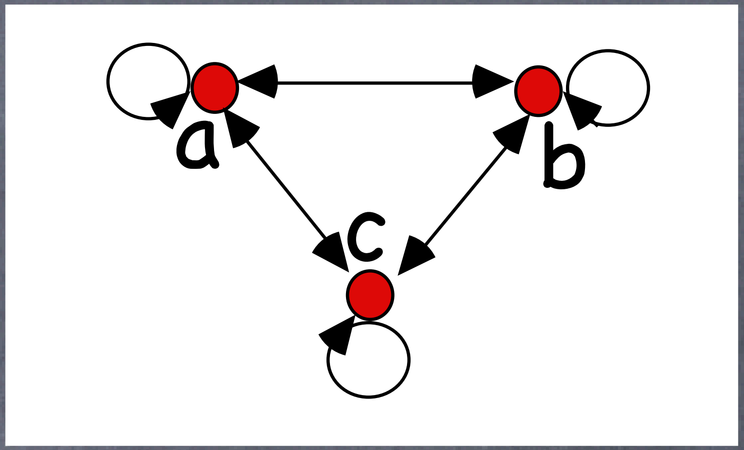
Equivalence Classes

- Let R be an equivalence relation on A. The set of all elements that are related to an element a of A is called the equivalence class of a, denoted by $[a]_R$ or [a] when only one relation is under consideration.
  - i.e. $[a]_R=\{s|(a,s)\in R\}$
  - a is called the representative of the equivalence class. We could have chosen any one in $[a]_R$

Examples







[a]={a},

[b]={b},

[c]={c}

[a]={a,b},

[b]={a,b},

[c]={c}

[a]={a,b,c},

[b]={a,b,c},

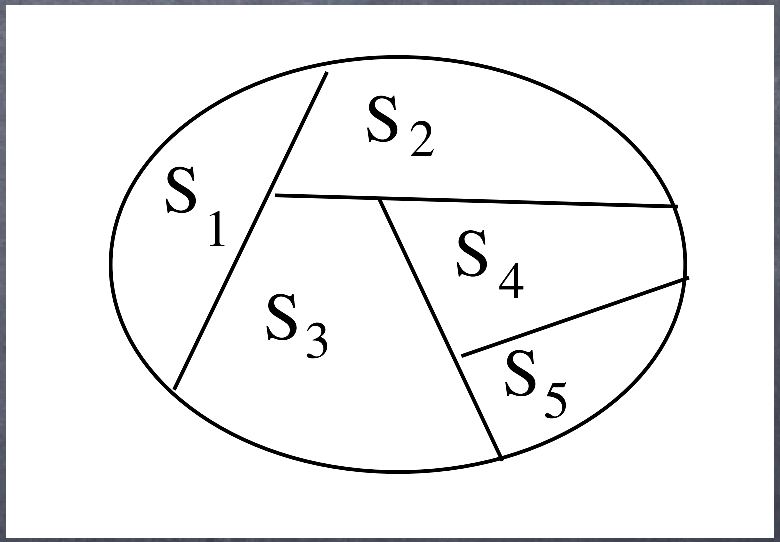
[c]={a,b,c}

Examples

- Let R be a relation on Z such that aRb iff a+b=0 (or a=-b). What is the equivalence class of an integer x?
  - If x=0, then [x]={0}; else [x]={x,-x}
- Let R be a relation on the set of bit strings of length three or more, such that aRb iff a and b agree in their first three bits. What is the equivalence class of 010?
  - [010] is the set of bit strings of length three or more that begin 010

Partition of Set

- Let $S_1, S_2, \ldots, S_n$ be a collection of subsets of A. Then the collection forms a partition of A if all subsets:
  - $S_i \neq \emptyset$
  - $S_i \cap S_j = \emptyset$ if $i \neq j$
  - $\cup S_i = A$



- E.g. $S=\{1,2,3,4,5,6\}$. The collection of sets $A_1=\{1,2,3\}$, $A_2=\{4,5\}$, $A_3=\{6\}$ forms a partition of S.

Equivalence Classes and Partitions

Let R be an equivalence relation on a set A.

- Theorem 1: the following statements for elements a and b of A are equivalent:
  - aRb
  - $[a]=[b]$
  - $[a]\cap[b]\neq\varnothing$
- Theorem 2: the equivalence classes of R form a partition of A.

Reading and Notes

- Recommended exercises: 8.5: 1,11,15,21,25,29,41,43