Relations and Their Properties

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Review

For two sets $A$ and $B$:

- Cartesian product $A \times B$
- Function from $A$ to $B$
- Binary relation from $A$ to $B$
A binary relation $R$ from a set $A$ to a set $B$ is a subset $R \subseteq A \times B$

- No restrictions on relations as on functions

Relations can be represented graphically: A directed graph $D$ from $A$ to $B$ is a collection of vertices $V \subseteq A \cup B$ and a collection of edges $R \subseteq A \times B$. If there is an ordered pair $e = \langle x, y \rangle$ in $R$ then there is an arc or edge from $x$ to $y$ in $D$. 

Binary Relation
Example: \( A = \{a,b,c\}, \ B = \{1,2,3,4\} \)

\( R = \{\langle a,1\rangle, \langle a,2\rangle, \langle c,4\rangle\} \)

Definition: A binary relation \( R \) on a set \( A \) is a subset of \( A \times A \) or a relation from \( A \) to \( A \).

Example:
- \( A = \{a, b, c\} \)
- \( R = \{\langle a, a\rangle, \langle a, b\rangle, \langle a, c\rangle\} \).

Then a digraph representation of \( R \) is:

\[\begin{array}{c}
\text{a} & \rightarrow & \text{1} \\
\text{b} & \rightarrow & \text{2} \\
\text{c} & \rightarrow & \text{3, 4}
\end{array}\]

Note: An arc of the form \( \langle x, x\rangle \) on a digraph is called a loop.

Question: How many binary relations are there on a set \( A \)?
A binary relation $R$ on a set $A$ is a subset of $A \times A$ or a relation from $A$ to $A$.

Eg. $A = \{a, b, c\}$, $R = \{<a, a>, <a, b>, <a, c>\}$

Q: How many binary relations are there on a set $A$?
Properties of Relations

Given a binary relation $R$ on a set $A$

- $R$ is reflexive iff $\forall x(x \in A \rightarrow <x,x> \in R)$
- $R$ is symmetric iff $\forall x \forall y (<x,y> \in R \rightarrow <y,x> \in R)$
- $R$ is antisymmetric iff $\forall x \forall y (<x,y> \in R \land <y,x> \in R \rightarrow x = y)$
- $R$ is transitive iff $\forall x \forall y \forall z$
  $(<x,y> \in R \land <y,z> \in R \rightarrow <x,z> \in R)$
A: not reflexive
symmetric
antisymmetric
transitive

B: not reflexive
not symmetric
not antisymmetric
not transitive

C: not reflexive
not symmetric
antisymmetric
not transitive

D: not reflexive
not symmetric
antisymmetric
transitive
Combining Relations

Two relations can be combined in any way two sets can be combined, using $U$, $\cap$ or $\setminus$.

Eg. $A=\{1,2,3\}$, $B=\{1,2,3,4\}$, $R_1=\{(1,1),(2,2),(3,3)\}$, $R_2=\{(1,1),(1,2),(1,3),(1,4)\}$, what is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

If $R_1$ and $R_2$ are transitive on $A$, does it follow that $R_1 \cup R_2$ is transitive?
Composition

Suppose \( R_1 \) is a relation from \( A \) to \( B \), \( R_2 \) is a relation from \( B \) to \( C \), then the composition of \( R_2 \) with \( R_1 \), denoted \( R_2 \circ R_1 \) is a relation from \( A \) to \( C \), such that if \( <a, b> \in R_1 \) and \( <b, c> \in R_2 \), then \( <a, c> \in R_2 \circ R_1 \).

\( R_2 \circ R_1 \) V.S. \( F_2 \circ F_1 \)

\[
R_2 \circ R_1 = \{<b, D>, <b, B>\}
\]
Let $R$ be a relation on set $A$, the powers $R^n$, $n=1,2,3\ldots$ are defined recursively by:

- $R^1 = R$
- $R^{n+1} = R^n \circ R$

Eg. Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find $R^n$, $n=2,3,4\ldots$
Theorem: \( R \) is transitive on a set \( A \) iff \( R^n \subseteq R \) for \( n > 0 \)

Proof:
1. \( R^n \subseteq R \) \( \rightarrow \) \( R \) is transitive

Assume \( R^n \subseteq R \). In particular, \( R^2 \subseteq R \).

If \( (a,b) \in R \), \( (b,c) \in R \), then \( (a,c) \in R^2 \). Therefore, \( (a,c) \in R \). So \( R \) is transitive.

2. \( R \) is transitive \( \rightarrow \) \( R^n \subseteq R \)

Basis: \( R^1 \subseteq R \)

Inductive step: Assume \( R^k \subseteq R \), need to show \( R^{k+1} \subseteq R \)

Because \( R^{k+1} = R^k \circ R \), if \( (a,b) \in R^{k+1} \) then there is an element \( x \in A \) such that \( (a,x) \in R \) and \( (x,b) \in R^k \). Because \( R^k \subseteq R \), \( (x,b) \in R \). Because \( R \) is transitive, \( (a,b) \in R \).
Reading and Notes

- Compare Relations and Functions
- Recommended exercises: 8.1: 3, 7, 35, 41, 43, 53