Relations and Their Properties

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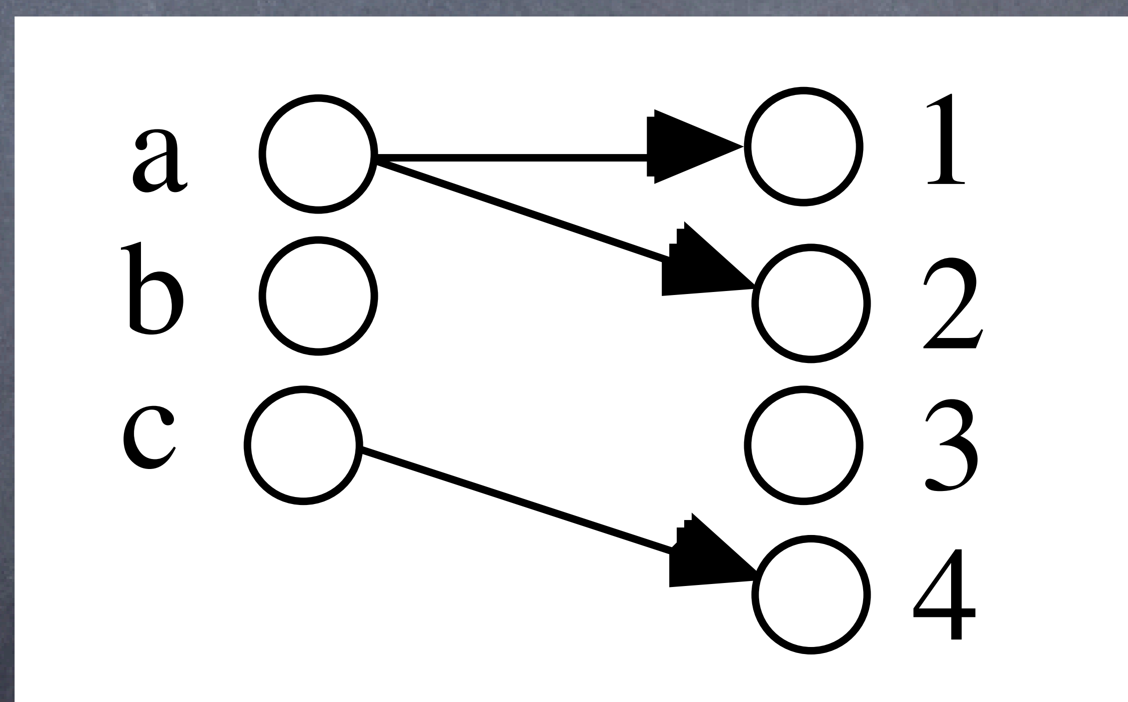
Review

- For two sets A and B:
  - Cartesian product AxB
  - Function from A to B
  - Binary relation from A to B

Binary Relation

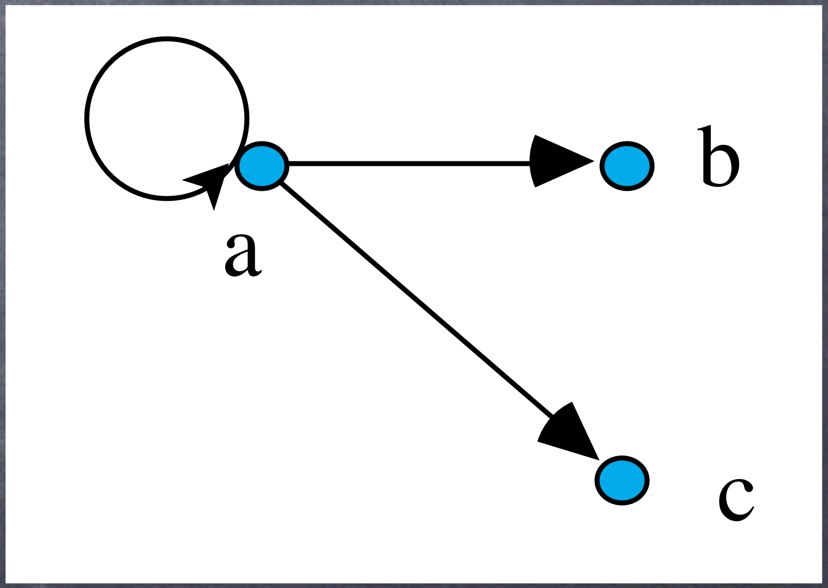
- A binary relation R from a set A to a set B is a subset $R \subseteq A \times B$
    - No restrictions on relations as on functions
- Relations can be represented graphically: A directed graph D from A to B is a collection of vertices $V \subseteq A \cup B$ and a collection of edges $R \subseteq A \times B$. If there is an ordered pair $e=\langle x,y \rangle$ in R then there is an arc or edge from x to y in D.

- Example: A={a,b,c}, B={1,2,3,4}
- R={<a,1>,<a,2>,<c,4>}



Relations on a Set

- A binary relation R on a set A is a subset of AxA or a relation from A to A
- Eg. A={a,b,c}, R={<a,a>,<a,b>,<a,c>}
- Q: How many binary relations are there on a set A?

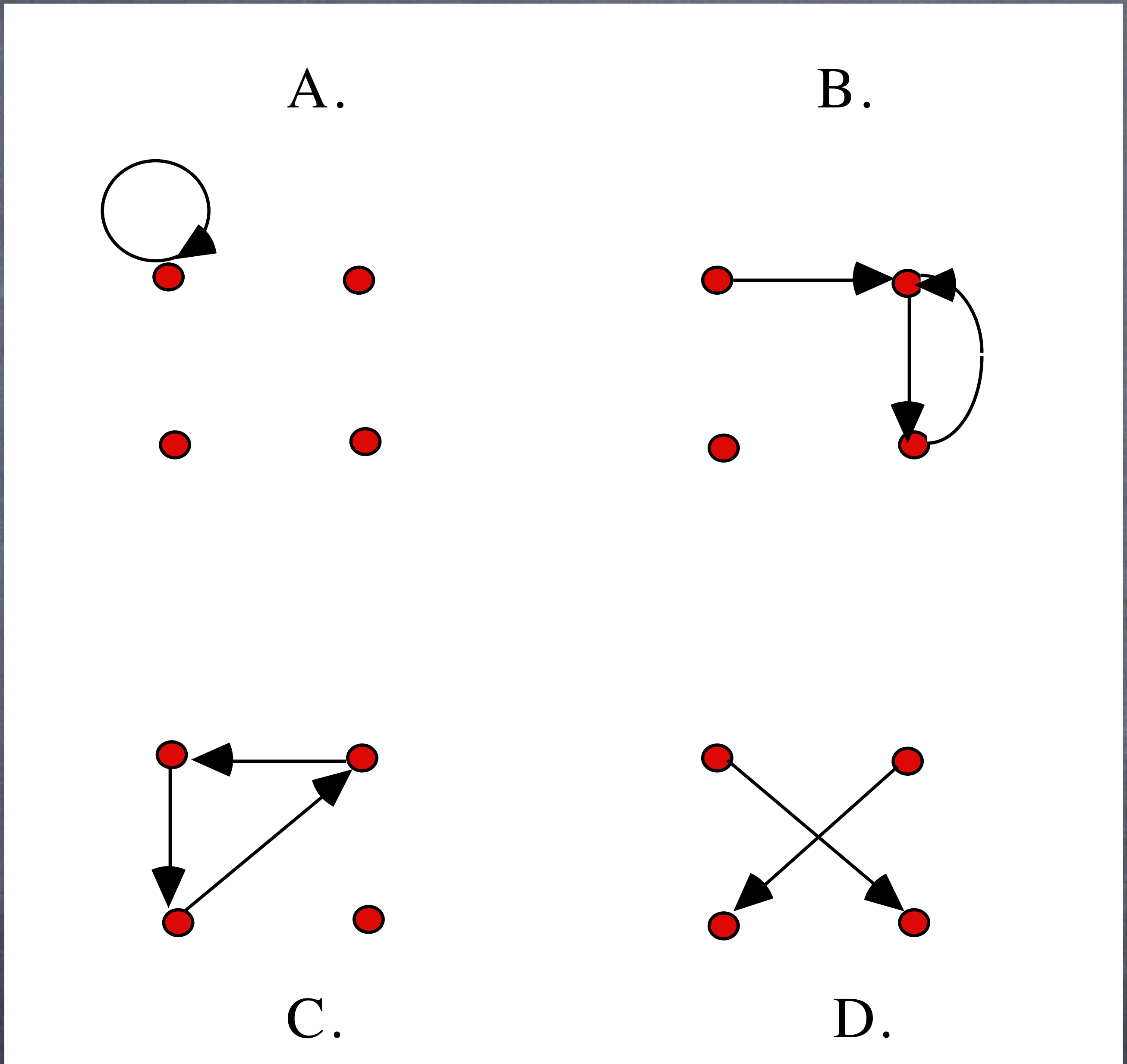


Properties of Relations

- Given a binary relation R on a set A
  - R is reflexive iff $\forall x(x \in A \rightarrow \langle x,x \rangle \in R)$
  - R is symmetric iff $\forall x \forall y(\langle x,y \rangle \in R \rightarrow \langle y,x \rangle \in R)$
  - R is antisymmetric iff $\forall x \forall y(\langle x,y \rangle \in R \wedge \langle y,x \rangle \in R \rightarrow x=y)$
  - R is transitive iff $\forall x \forall y \forall z$ $(\langle x,y \rangle \in R \wedge \langle y,z \rangle \in R \rightarrow \langle x,z \rangle \in R)$

A:
not reflexive
symmetric
antisymmetric
transitive

B:
not reflexive
not symmetric
not antisymmetric
not transitive



C:
not reflexive
not symmetric
antisymmetric
not transitive

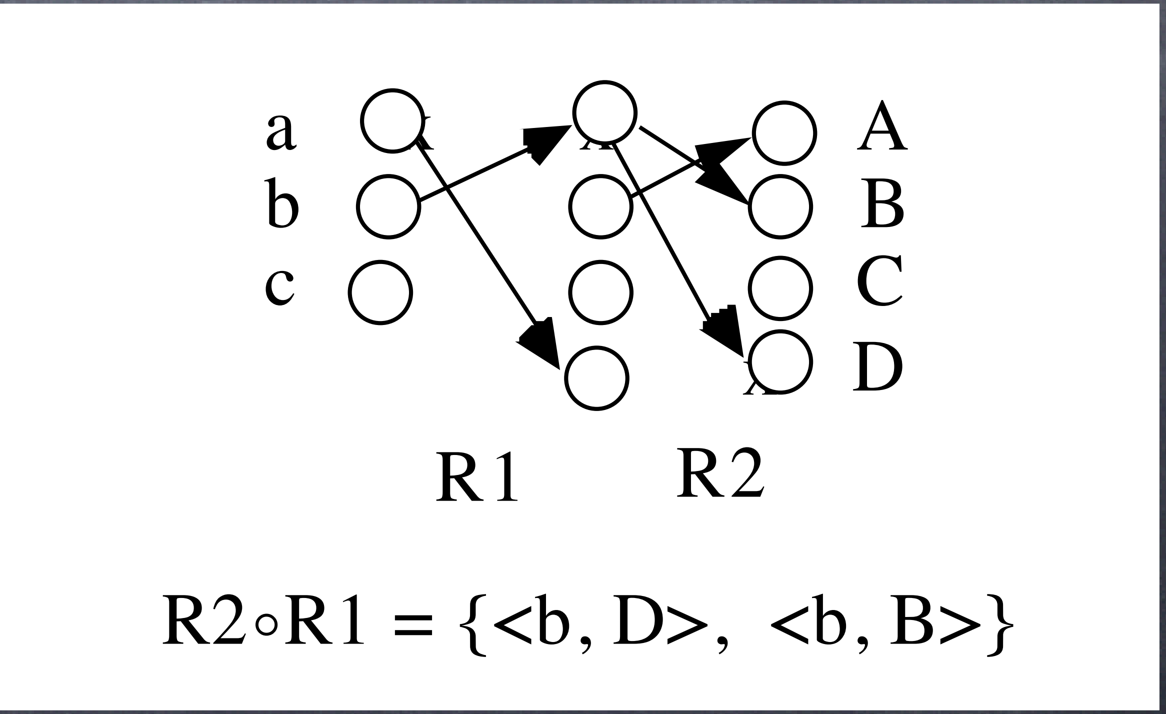
D:
not reflexive
not symmetric
antisymmetric
transitive

Combining Relations

- Two relations can be combined in any way two sets can be combined, using ∪, ∩ or -.
- Eg. A={1,2,3}, B={1,2,3,4}, R1={(1,1),(2,2),(3,3)}, R2={(1,1),(1,2),(1,3),(1,4)}, what is R1∪R2, R1∩R2, R1-R2, R2-R1?
- If R1 and R2 are transitive on A, does it follow that R1∪R2 is transitive?

Composition

- Suppose R1 is a relation from A to B, R2 is a relation from B to C, then the composition of R2 with R1, denoted $R2 \circ R1$ is a relation from A to C, such that if $\langle a,b \rangle \in R1$ and $\langle b,c \rangle \in R2$, then $\langle a,c \rangle \in R2 \circ R1$
- $R2 \circ R1$ V.S. $F2 \circ F1$



Powers R^n

- Let R be a relation on set A, the powers R^n, n=1,2,3... are defined recursively by:
  - $R^1=R$
  - $R^{n+1}=R^n \circ R$
- Eg. Let R={(1,1),(2,1),(3,2),(4,3)}. Find R^n, n=2,3,4...

Theorem: R is transitive on a set A iff $R^n \subseteq R$ for $n>0$

Proof:

1. $R^n \subseteq R$ -> R is transitive

Assume $R^n \subseteq R$. In particular, $R^2 \subseteq R$.

If $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R^2$. Therefore, $(a,c) \in R$. So R is transitive

2. R is transitive -> $R^n \subseteq R$

Basis: $R^1 \subseteq R$

Inductive step: Assume $R^k \subseteq R$, need to show $R^{k+1} \subseteq R$

Because $R^{k+1} = R^k \circ R$, if $(a,b) \in R^{k+1}$ then there is an element $x \in A$ such that $(a,x) \in R$ and $(x,b) \in R^k$. Because $R^k \subseteq R$, $(x,b) \in R$. Because R is transitive, $(a,b) \in R$.

Reading and Notes

- Compare Relations and Functions
- Recommended exercises: 8.1: 3,7,35,41,43,53