Solving Linear Recurrence Relations

Jing Yang

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Overview

- Solve recurrence relations: find a formula for $\{a_n\}$
- Easy: for $a_n=2a_{n-1}$, $a_0=1$, the solution is $a_n=2^n$ (back substitute)
- Difficult: for $a_n=a_{n-1}+a_{n-2}$, $a_0=0$, $a_1=1$, how to find a solution?

Linear Homogeneous Recurrence Relations of degree k with constant coefficients

- Solving a recurrence relation can be very difficult unless the recurrence equation has a special form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ where $c_1, c_2, \ldots c_k \in R$ and $c_k \neq 0$

  - Single variable: n
  - Linear: no $a_i a_j$, $a_i^2, a_i^3 \ldots$
  - Constant coefficients: $c_i \in R$
  - Homogeneous: all terms are multiples of the a_is
  - Degree k: $c_k \neq 0$

Example

- ✓ $a_n=1.02a_{n-1}+a_{n-2}$

  linear, constant coefficients, homogeneous, degree 2
- ✓ $a_n=1.02a_{n-3}$

  linear, constant coefficients, homogeneous, degree 3
- – $a_n=1.02a_{n-1}+2^{n-1}$

  linear, constant coefficients, nonhomogeneous, degree 1
- – $a_n=a_{n-1}+a_{n-2}+a_{n-3}+2^{n-1}$

  linear, constant coefficients, nonhomogeneous, degree 3
- – $a_n=ca_{n+m}$

  two variables
- – $a_n=na_{n-1}+n^2a_{n-2}$

  linear, coefficients are not constant, homogeneous, degree 2
- – $a_n=a_{n-1}a_{n-2}$

  nonlinear, constant coefficients, homogeneous, degree 2

Solution Procedure

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ where $c_1, c_2, \ldots c_k \in R$ and $c_k \neq 0$

1. Put all a_i's on LHS of the equation: $a_n - c_1 a_{n-1} - c_2 a_{n-2} - \ldots - c_k a_{n-k} = 0$
2. Assume solutions of the form $a_n = r^n$, where r is a constant
3. Substitute the solution into the equation:

   $r^n - c_1 r^{n-1} - c_2 r^{n-2} \ldots - c_k r^{n-k} = 0$. Factor out the lowest power of r:

   $r^k - c_1 r^{k-1} - c_2 r^{k-2} \ldots - c_k = 0$ -- called the characteristic equation
4. Find the k solutions $r_1, r_2, \ldots, r_k$ of the characteristic equation (characteristic roots of the recurrence relation)
5. If the roots are distinct, the general solution is

   $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$ Theorem!
6. The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ are found by enforcing the initial conditions

Example - 1

Solve $a_{n+2}=3a_{n+1}$, $a_0=4$

- $a_{n+2}-3a_{n+1}=0$
- $r^{n+2}-3r^{n+1}=0$, i.e. $r-3=0$
- Find the root of the characteristic equation $r_1=3$
- Compute the general solution $a_n=\alpha_1 3^n$
- Find α_1 based on the initial conditions: $a_0=\alpha_1(3^0)$ or $\alpha_1=4$
- Produce the solution: $a_n=4(3^n)$

Example - 2

Solve $a_n=3a_{n-2}$, $a_0=a_1=1$

- $a_n-3a_{n-2}=0$
- $r^n-3r^{n-2}=0$, i.e. $r^2-3=0$
- Find the root of the characteristic equation $r_1=\surd 3$, $r_2=-\surd 3$
- Compute the general solution $a_n=\alpha_1(\surd 3)^n+\alpha_2(-\surd 3)^n$
- Find α_1 and α_2 based on the initial conditions:

  $a_0=\alpha_1(\surd 3)^0+\alpha_2(-\surd 3)^0=\alpha_1+\alpha_2=1$

  $a_1=\alpha_1(\surd 3)^1+\alpha_2(-\surd 3)^1=\surd 3\alpha_1-\surd 3\alpha_2=1$
- Solution: $a_n=(1/2+1/2\surd 3)(\surd 3)^n+(1/2-1/2\surd 3)(-\surd 3)^n$

Example - 3

Find an explicit formula for the Fibonacci numbers

- $f_n - f_{n-1} - f_{n-2} = 0$
- $r^n - r^{n-1} - r^{n-2} = 0$, i.e. $r^2 - r - 1 = 0$
- Find the root of the characteristic equation $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$
- Compute the general solution $f_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n$
- Find α_1 and α_2 based on the initial conditions:

  $\alpha_1 = 1/\sqrt{5}$

  $\alpha_2 = -1/\sqrt{5}$
- Solution: $f_n = 1/\sqrt{5} \cdot ((1+\sqrt{5})/2)^n - 1/\sqrt{5} \cdot ((1-\sqrt{5})/2)^n$

Example - 4

Find the solution to the recurrence relation $a_n=6a_{n-1}-11a_{n-2}+6a_{n-3}$ with $a_0=2$, $a_1=5$, $a_2=15$

- $a_n-6a_{n-1}+11a_{n-2}-6a_{n-3}=0$
- $r^n-6r^{n-1}+11r^{n-2}-6r^{n-3}=0$, i.e. $r^3-6r^2+11r-6=0$
- Find the root of the characteristic equation $(r-1)(r-2)(r-3)=0$: $r_1=1$, $r_2=2$, $r_3=3$
- Compute the general solution $f_n=\alpha_1(r_1)^n+\alpha_2(r_2)^n+\alpha_3(r_3)^n$
- Find α_1, α_2 and α_3 based on the initial conditions
- See solution in textbook

If a root r_1 has multiplicity m_1 then the solution is
$a_n=\alpha_1 r_1{}^n+\alpha_2 n r_1{}^n+\ldots+\alpha_{m1} n^{m1-1} r_1{}^n+\ldots$

Eg. Solve $a_n=6a_{n-1}-9a_{n-2}$, $a_0=a_1=1$

- $a_n-6a_{n-1}+9a_{n-2}=0$
- $r^n-6r^{n-1}+9r^{n-2}=0$, i.e. $r^2-6r+9=0$
- Root of the characteristic equation $r_1=r_2=3$
- General solution $a_n=\alpha_1 3^n+\alpha_2 n 3^n$
- Solve for the coefficients:

  $a_0=\alpha_1+0=1$, so $\alpha_1=1$

  $a_1=1(3)^1+\alpha_2(1)(3)^1=1$, so $\alpha_2=-2/3$
- $an=3^n-2/3 \cdot n \cdot 3^n$

Reading and Notes

- Master the solution procedure for linear homogeneous recurrence relations with constant coefficients
- Recommended exercises: 7.2: 1,3,15,21