Recurrence Relations

Jing Yang November 19, 2010

1

Overview

An easy counting problem: How many bit strings of length n have exactly three zeros?

A more difficult counting problem: How many bit strings of length n contain three consecutive zeros?

Recursive definition and recurrence relation

Recurrence Relations

A recurrence relation (sometimes called a difference equation) is an <u>equation</u> which defines the nth term in the sequence in terms of (one ore more) previous terms

A sequence is called a <u>solution</u> of a recurrence relation if its terms satisfy the recurrence relation

Modeling with Recurrence Relations

Seamples:

□ Fibonacci sequence: a_n=a_{n-1}+a_{n-2}

□ Pascal's identity: C(n+1,k)=C(n,k)+C(n,k-1)

Normally there are infinitely many sequences which satisfy the equation. These solutions are distinguished by the initial conditions.

 \square Eg. Find the solution to $a_n=a_{n-1}+n$, $a_0=1$

Example 1 – Easy

Suppose interest is compounded at 11% annually. If we deposit \$10,000 and do not withdraw the interest, find the total amount invested after 30 years.

- Recurrence relation: $P_n = P_{n-1} + 0.11P_{n-1}$
- Initial condition: $a_0=10,000$
- \odot Answer: P₃₀=10000x(1.11)³⁰

Example 2 (harder)

Find a recurrence relation for the number of bit strings of length n that do not have two consecutive Os.

- an: # strings of length n that do not have two consecutive Os.
- \square Case 1: # strings of length n ending with 1 -- a_{n-1}
- \square Case 2: # strings of length n ending with 10 -- a_{n-2}
- □ This yields the recurrence relation a_n=a_{n-1}+a_{n-2} for n≥3

6

 \square Initial conditions: $a_1=2$, $a_2=3$

Example 3 (much harder)

Find a recurrence relation for the number of bit strings of length n which contain 3 consecutive Os.

- Let S be the set of strings with 3 consecutive Os.
 First define the set inductively.
 - 🗆 Basis: 000 is in S
 - □ Induction (1): if w∈S, u∈{0,1}*, v∈{0,1}* then uwv∈S

Adequate to define S but NOT for counting. DO NOT count the same string twice.

Example 3 (much harder)

Find a recurrence relation for the number of bit strings of length n which contain 3 consecutive Os.

- Let S be the set of strings with 3 consecutive Os. First define <u>the set</u> inductively.
 - Induction (2): if $w \in S$, $u \in \{0,1\}^*$, then
 - $1w \in S$, $01w \in S$, $001w \in S$, $000u \in S$
- \square This yields the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$
- \square Initial conditions: $a_3=1, a_4=3, a_5=8$

Reading and Notes

Recommended exercises: 7.1:1,9,11,13,15,19