Binomial Coefficients

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Overview

- P(n,r) vs C(n,r)
- Other names for C(n,r):
  - n choose r
  - the binomial coefficient

How many bit strings of length 10 contain

- exactly four 1s?

$C(10,4)=210$

- at most three 1s?

$C(10,0)+C(10,1)+C(10,2)+C(10,3)=176$

- at least 4 1s?

$2^{10}-176=848$

- an equal number of 0s and 1s?

$C(10,5)=252$

Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j \text{ for } n \geq 0$$

- $C(n,r)$ occurs as coefficients in the expansion of $(a+b)^n$
- Combinatorial proof: refer to textbook

Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j \text{ for } n \geq 0$$

- Examples:
  - What is the expansion of $(x+y)^4$?

    $$\boxed{x^4+4x^3y+6x^2y^2+4xy^3+y^4}$$

  - What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

    $$\boxed{-(25!\times 2^{12}\times 3^{13})/(13!12!)}$$

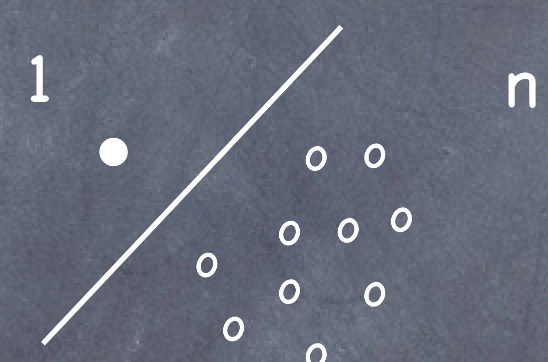
Corollary

$$\sum_{j=0}^{n} C(n,j) = 2^n \text{ for } n \geq 0$$

- Proof 1: Use the Binomial Theorem with x=y=1
- Proof 2 (combinatorial proof): Use the subsets, and the cardinality of the powerset.

Pascal's Identity

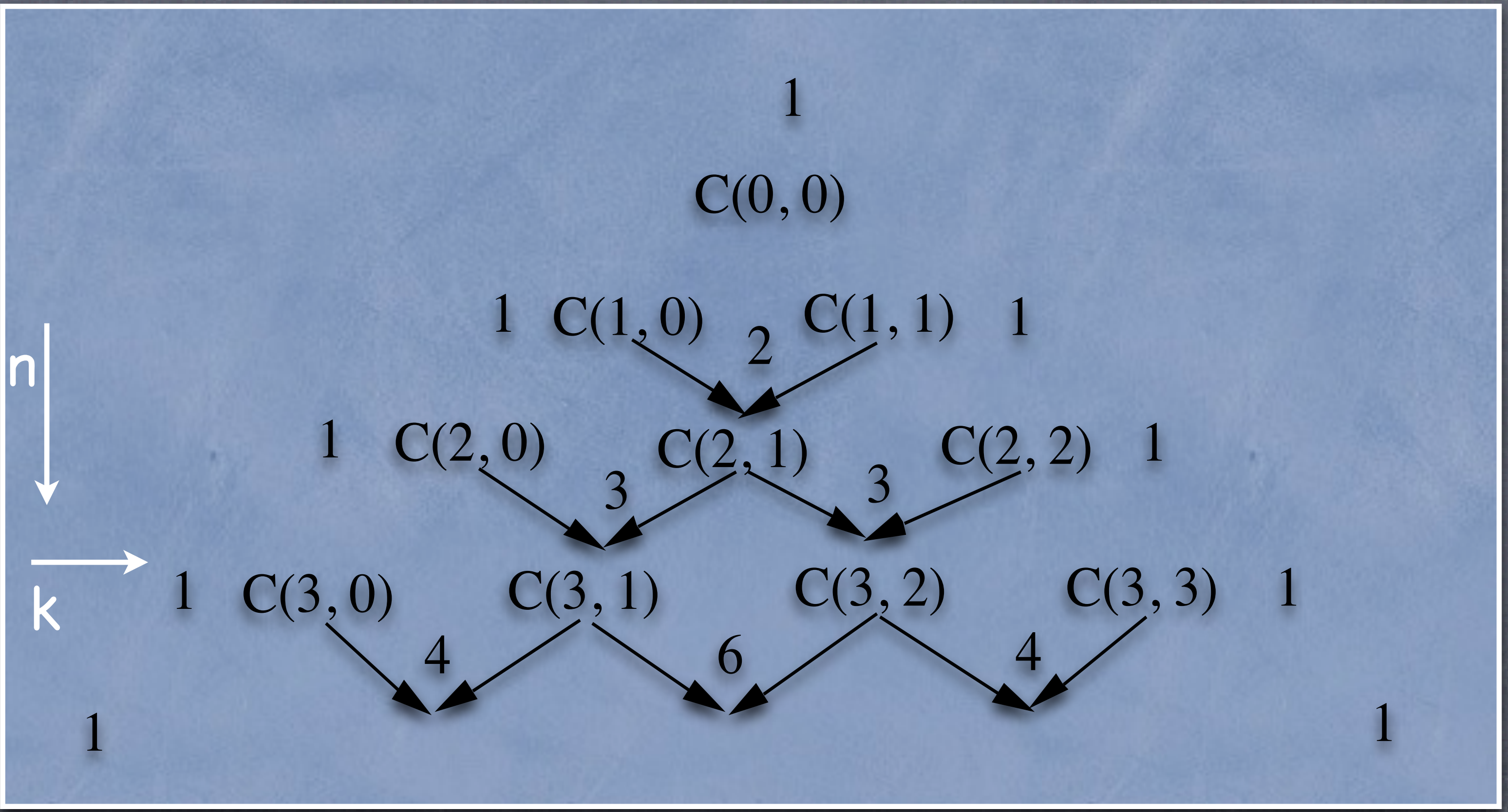
$$C(n+1,k) = C(n,k-1) + C(n,k) \text{ for } n \geq k \geq 1$$



Total number of subsets = number including • + number not including •

$C(n+1,k)$ $C(n,k-1)$ $C(n,k)$

Pascal's triangle



A good way to evaluate C(n,r) for large n and r.

Reading and Notes

- Understand the rationale behind the equivalence of C(n,r) and binomial coefficients.
- Recommended exercises: 5.4:1,5,9,11