Recursive Definitions

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Overview

- Geometric progression: $a_n=ar^n$ for $n=0,1,2,\ldots$
  - A recursive definition: $a_0=a$, $a_n=ra_{n-1}$ for $n=1,2,3\ldots$
- Arithmetic progression: $a_n=a+dn$ for $n=0,1,2,\ldots$
  - A recursive definition: $a_0=a$, $a_n=a_{n-1}+d$ for $n=1,2,3\ldots$
- Recursive definition: define an object by **itself**

Recursive Definition

1. Basis step
    - For functions: specify the value of the function at zero
    - For sets: specify members of the initial set
2. Inductive or recursive step
    - For functions: Show how to compute its value at an integer from its values at smaller integers.
    - For sets: Show how to build new things from old things with some construction rules

Practice

- Suppose $f(0)=3$, and $f(n+1)=2f(n)+2$, $\forall n \geq 0$. Find $f(1)$, $f(2)$ and $f(3)$.
- Give a recursive definition for $f(n)=n!$
- Give a recursive definition for $f(n)=2$

Fibonacci Sequence

- Recursive definition:

  1. Basis:

     $f(0)=0$, $f(1)=1$ (two initial conditions)

  2. Induction:

     $f(n)=f(n-1)+f(n-2)$ for $n=2,3,4\ldots$ (recurrence equation)

- Practice: find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6

Recursive Definition & Mathematical Induction

Proof of assertions about recursively defined objects usually involves a proof by induction.

- Prove the assertion is true for the basis step
- Prove if the assertion is true for the old objects it must be true for the new objects you can build from the old objects
- Conclude the assertion must be true for all objects

For fibonacci numbers prove that $f_n > \alpha^{n-2}$ when $n \geq 3$, where $\alpha = (1+\sqrt{5})/2$.

Proof by strong induction: $P(n)$ is $f_n > \alpha^{n-2}$

Basis step: $\alpha < 2 = f_3$, $\alpha^2 = (3+\sqrt{5})/2 < 3 = f_4$. So $P(3)$ and $P(4)$ are true

Inductive step: Assume that $P(j)$ is true, i.e. $f_j > \alpha^{j-2}$ for $3 \leq j \leq k$, where $k \geq 4$. We must show $P(k+1)$ is true.

$$f_{k+1} = f_k + f_{k-1} > \alpha^{k-2} + \alpha^{k-3} = (\alpha+1)\alpha^{k-3} = \alpha^2\alpha^{k-3} = \alpha^{k-1}$$

$\alpha + 1 = \alpha^2$

Practice

Give a recursive definition for the following sets:

- Z^+
- The set of odd positive numbers
- The set of positive numbers not divisible by 3

Strings

- A string over an alphabet Σ is a finite sequence of symbols from Σ

- The set of all strings (including the empty string λ) is called Σ^*

- Recursive definition of Σ^*:

  1. Basis step: $\lambda \in \Sigma^*$

  2. Recursive step: If $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$

- Example: If $\Sigma = \{0,1\}$, then $\Sigma^* = \{\lambda,0,1,00,01,10,11,\ldots\}$ is the set of bit strings

Give a recursive definition of the set S of bit strings with no more than a single 1.

1. Basis step: λ, 0, and 1 are in S

2. Recursive step: if w is in S, then so are 0w and w0.

Length of a String

- Recursive definition of the length of a string $l(w)$

1. Basis step: $l(\lambda)=0$
2. Inductive step: $l(wx)=l(w)+1$ if $w\in\Sigma^*$ and $x\in\Sigma$

Reading and Notes

- Understand the recursive definitions for functions and sets
- Practice transforming the recursive definition to a definition by formula, and vice versa
- Recommended exercises: 4.3:9,13,23,45,57,59,61,62