Mathematical Induction

Jing Yang

November 1 2010

Overview

- Proof methods: direct proof, proof by cases, proof by contraposition, proof by contradiction, disproof by counterexample
- Mathematical induction is often used to prove P(x) is true for all positive integers

Principle of Mathematical Induction

$$(P(1) \wedge \forall k(P(k) \rightarrow P(k+1)) \rightarrow \forall n P(n)$$

- To prove that $\forall n P(n)$, where $n \in Z^+$ and $P(n)$ is a propositional function, we complete two steps:

  i) Basis step: Verify $P(1)$ is true

  ii) Inductive step: Show $P(k) \rightarrow P(k+1)$ is true for arbitrary $k \in Z^+$

Mathematical Induction

- Knowing it is true for the first element means it must be true for the next element, i.e. the second element
- Knowing it is true for the second element implies it is true for the third

and so forth.

Mathematical Induction

- How to show P(1) is true?
  - Replace n by 1 in P(n)
- How to show P(k)→P(k+1) is true?
  - Direct proof is normally used
  - Assume P(k) is true for some arbitrary k
  - Then show P(k+1) is true

Inductive Hypothesis

Show that $1+2+3+\ldots+n = n(n+1)/2$, where $n \in Z^+$

Proof by induction:

First identify P(n), P(n): $1+2+\ldots.+n=n(n+1)/2$

Basis step: P(1): $1 = 1 \times 2/2 = 1$. So, P(1) is true

Inductive step: Assume P(k) is true for arbitrary $k \in Z^+$, i.e. $1+2+\ldots+k=k(k+1)/2$.

Need to show P(k+1): $1+2+\ldots.+k+k+1 = (k+1)(k+2)/2$ is true.

LHS = $(1+2+\ldots+k)+k+1 = k(k+1)/2+k+1 = (k+1)(k+2)/2$

So LHS = RHS. We showed P(k+1) is true under the assumption that P(k) is true.

By mathematical induction P(n) is true for all positive integers

Mathematical Induction (2)

- How to show that P(n) is true for n=b, b+1, b+2,.... where b is an integer other than 1?

i) Basis step: Verify P(b) is true

ii) Inductive step: Show $P(k) \rightarrow P(k+1)$ is true for arbitrary $k \in Z$

Show that $1+2+2^2+...+2^n = 2^{n+1}-1$, where $n \in N$

Proof by induction:

First identify P(n), P(n): $1+2+2^2.... +2^n = 2^{n+1}-1$

Basis step: P(0): $1 = 2^1-1 = 1$. So, P(0) is true

Inductive step: Assume P(k) is true for arbitrary $k \in N$, i.e. $1+2+...+2^k = 2^{k+1}-1$

Need to show P(k+1): $1+2+....+2^k+2^{k+1} = 2^{k+2}-1$ is true.

LHS = $(1+2+...+2^k)+2^{k+1} = 2^{k+1}-1+2^{k+1} = 2^{k+2}-1$

So LHS = RHS. We showed P(k+1) is true under the assumption that P(k) is true.

By mathematical induction P(n) is true for all natural numbers

Prove: $\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r-1} \ \ if\ r \neq 1$

Proof by induction:

First identify P(n), P(n): $a+ar+ar^2 .... +ar^n = (ar^{n+1}-a)/(r-1)$

Basis step: P(0): $a = (ar-a)/(r-1)$. So, P(0) is true

Inductive step: Assume P(k) is true for arbitrary $k \in N$, i.e. $a +ar+ar^2 .... +ar^k = (ar^{k+1}-a)/(r-1)$

Need to show P(k+1): $a+ar+.... +ar^k+ar^{k+1}=(ar^{k+2}-a)/(r-1)$ is true.

LHS = $(ar^{k+1}-a)/(r-1) +ar^{k+1} = (ar^{k+1}-a +ar^{k+2} -ar^{k+1})/(r-1)$

So LHS = RHS. We showed P(k+1) is true under the assumption that P(k) is true.

By mathematical induction P(n) is true for all natural numbers

Prove $2^n < n!$ for every positive integer n with $n > 3$

Proof by induction: in the book