The Growth of Functions

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Overview

- How fast does a function grow? How to measure it?
- We quantify the concept that g grows at least as fast as f.
- What really matters in comparing the complexity of algorithms?
  - We only care about the behavior for large problems
  - Even bad algorithms can be used to solve small problems

Big-O Notation

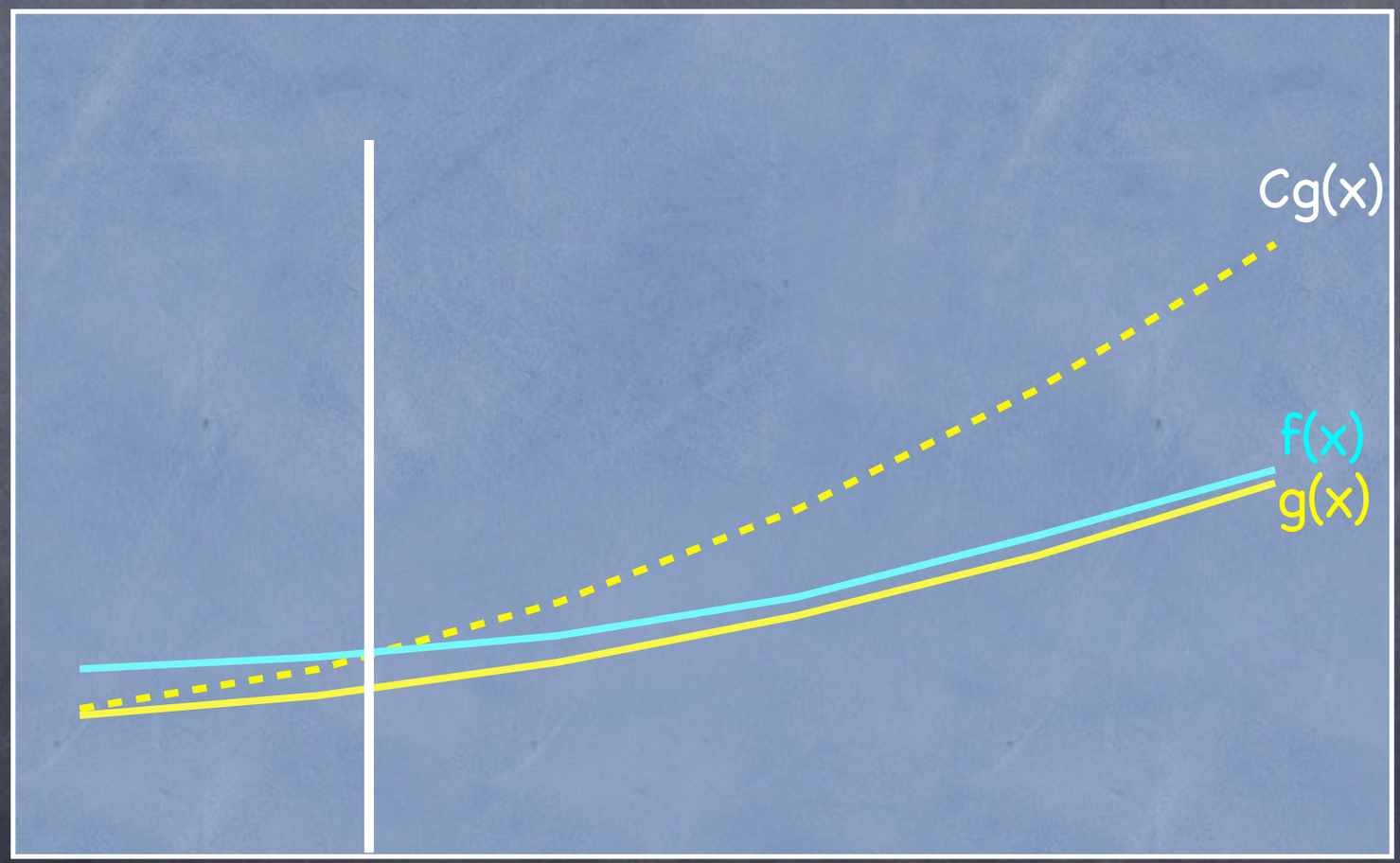
Assume $f: Z \to R$ and $g: Z \to R$.

$f(x)$ is $O(g(x))$ iff $\exists$ constants C and k such that

$$\forall x > k \; |f(x)| \leq C|g(x)|$$

- Constants C and k are called witnesses
- The choice of C may depend on the choice of k
- When there is one pair of witnesses, there are infinitely many pairs of witnesses

Big-O Notation



Big-O Notation

- O(g) is a set called a complexity class
- O(g) contains all the functions which g dominates
- f is O(g) or f=O(g) means f∈O(g)

Big-O notation (example)

$f(x) = x^2+2x+1$ is $O(x^2)$.

Proof: Observe that whenever $x>1$, $1<x<x^2$ is true.

Then it follows that for $x>1$

$0 \leq x^2+2x+1 = |f(x)| \leq x^2+2x^2+x^2=4x^2 = 4|x^2|$

$\therefore k=1$ and $C=4$

$\therefore f(x) = O(x^2)$ or $f(x) \in O(x^2)$

Big-O notation (example)

$f(x) = 7x^2$ is $O(x^3)$.

Proof: Observe that whenever $x>1$, $x^2<x^3$ is true.

Then it follows that for $x>1$

$0 \leq 7x^2 = |f(x)| \leq 7x^3 = 7|x^3|$

$\therefore k=1$ and $C=7$

$\therefore f(x) = O(x^3)$ or $f(x) \in O(x^3)$

Big-O notation (example)

Is it true that x^3 is $O(7x^2)$?

Determine whether witnesses exist or not.

Assume we can find C and k such that

$x^3 \leq C(7x^2)$ whenever $x>k$

i.e. $x \leq 7C$ whenever $x>k$

No matter what C and k are, the inequality $x \leq 7C$ cannot hold for all x with $x>k$.

So, x^3 is not $O(7x^2)$.

Growth of polynomial functions

- The leading term of a polynomial function determines its growth
- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where a_n, a_{n-1}, …, a_1, a_0 are real numbers. Then $f(x)$ is $\mathbf{O(x^n)}$.
- See the proof in textbook