Sequences & Summations (2)

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Double Summation

$$
\begin{aligned}
&\sum_{i=1}^{4}\sum_{j=1}^{3} ij \\
&= \sum_{i=1}^{4}(i + 2i + 3i) \\
&= \sum_{i=1}^{4} 6i \\
&= 6 + 12 + 18 + 24 = 60
\end{aligned}
$$

loop 1: for i=1 to 4

loop 2: for j=1 to 3

S = S + ij

Cardinality

- Recall: A set is finite if its cardinality is some (finite) integer n
- For two sets A and B
  - $|A| = |B|$ if and only if there is a bijection from A to B
  - $|A| \leq |B|$ if there is an injection from A to B
  - $|A| = |B|$ if $|A| \leq |B|$ and $|B| \leq |A|$
  - $|A| \leq |B|$ if $A \subseteq B$

Cardinality - 2

- Cardinality of infinite sets
  - Why do we care?
  - Do all the infinite sets have the same cardinality?
- Note: With infinite sets proper subsets can have the same cardinality.
  - This cannot happen with finite sets.

Countability - 1

- A set is countable if
  - it is finite or
  - it has the same cardinality as the set of the positive integers Z^+, i.e. $|A| = |Z^+|$. The set is countably infinite
- We write $|A| = |Z^+| = \aleph_0$ = aleph null
- A set that is not countable is called uncountable

Countability - 2

- Countability implies that there is a listing of the elements of the set.
- Proving the set is countable involves (usually) constructing an explicit bijection with Z^+

Examples

- {0,2,5} -- countable
- The set of odd integers -- countably infinite

Show that the set of odd positive integers S is countable.

Solution: To show that S is countable, we will show a bijective function between Z^+ and S.

Consider f: Z^+->S be such that $f(n) = 2n-1$.

To see f is one-to-one, suppose that $f(n)=f(m)$, then $2n-1=2m-1$, so $n=m$.

To see f is onto, suppose that $t \in S$, i.e. $t=2k-1$ for some positive integer k. Hence $t=f(k)$.

Q.E.D.

- Obvious fact: Any subset of a countable set is countable
- Less obvious facts
  - The rationals are countable (will prove)
  - The reals are not countable (in the book)

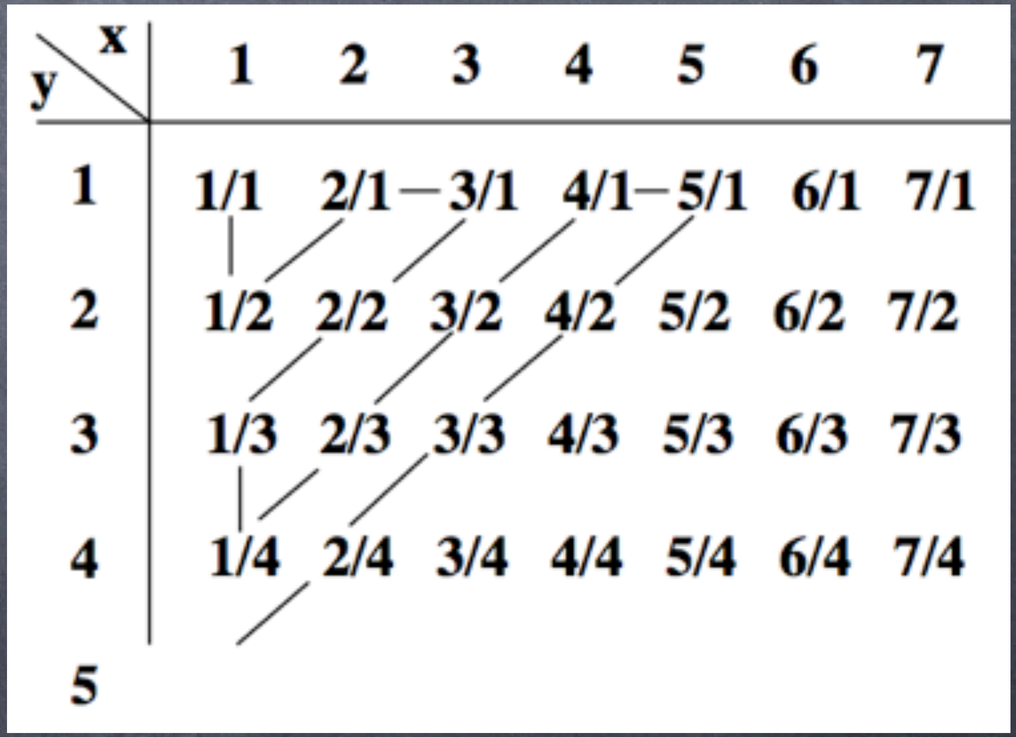
Prove that the set of positive rational numbers Q^+ is countable

Proof: Z^+ is a subset of Q^+, so $|Z^+| = \aleph_0 \leq |Q^+|$.

Now need to show that the positive rational numbers with repetitions, Q_R is countably infinite (*)

Since $Q^+ \subseteq Q_R$, $|Q^+| \leq |Q_R| = \aleph_0$. Hence $|Q^+| = \aleph_0$

Subproof: show (*)



The position on the path (listing) indicates the image of the bijective function from Z^+ to Q^R:

f(1)=1/1, f(2)=1/2, f(3)=2/1, f(4)=3/1

Every rational number appears on the list at least once, some any times.

Reading and Notes

- Understand the relationship between Sets, Functions, Sequences, and Set Countability
- Practice finding the value of sums of sequences
- Practice determining the countability of a set
- Recommended exercises: 2.4: 5,13,19,31,35