Functions

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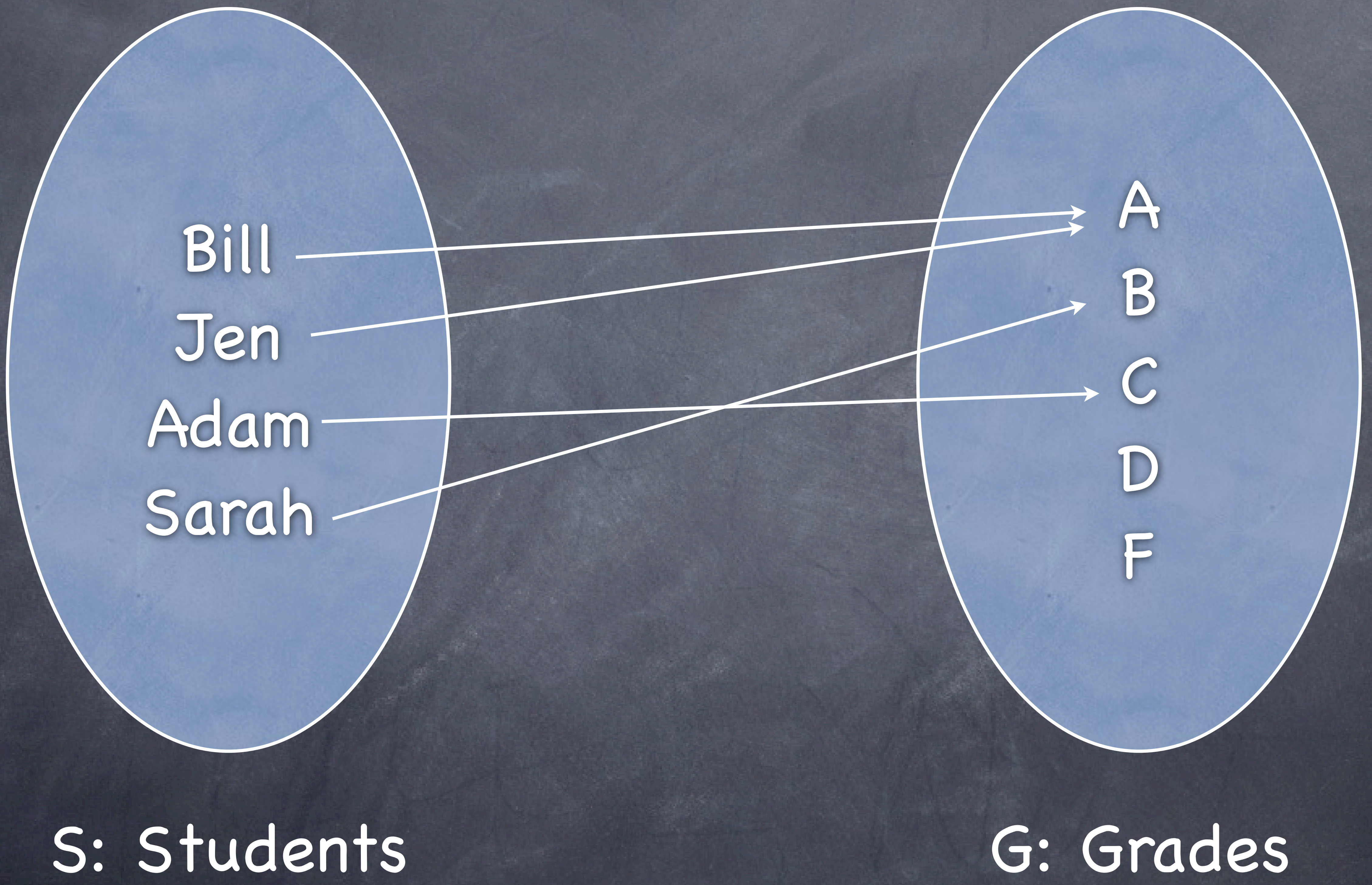

Bill
Jen
Adam
Sarah
A
B
C
D
F
S: Students
G: Grades

Definitions

- A function f is a mapping from a set A to a set B with the property that **for each** x in A, the function f maps x to a **single** element of B, denoted f(x).
- Formal definition:

$$f: A \rightarrow B$$

is a subset of AxB such that

$$\forall x(x \in A \rightarrow \exists y(y \in B \wedge \langle x,y \rangle \in f))$$

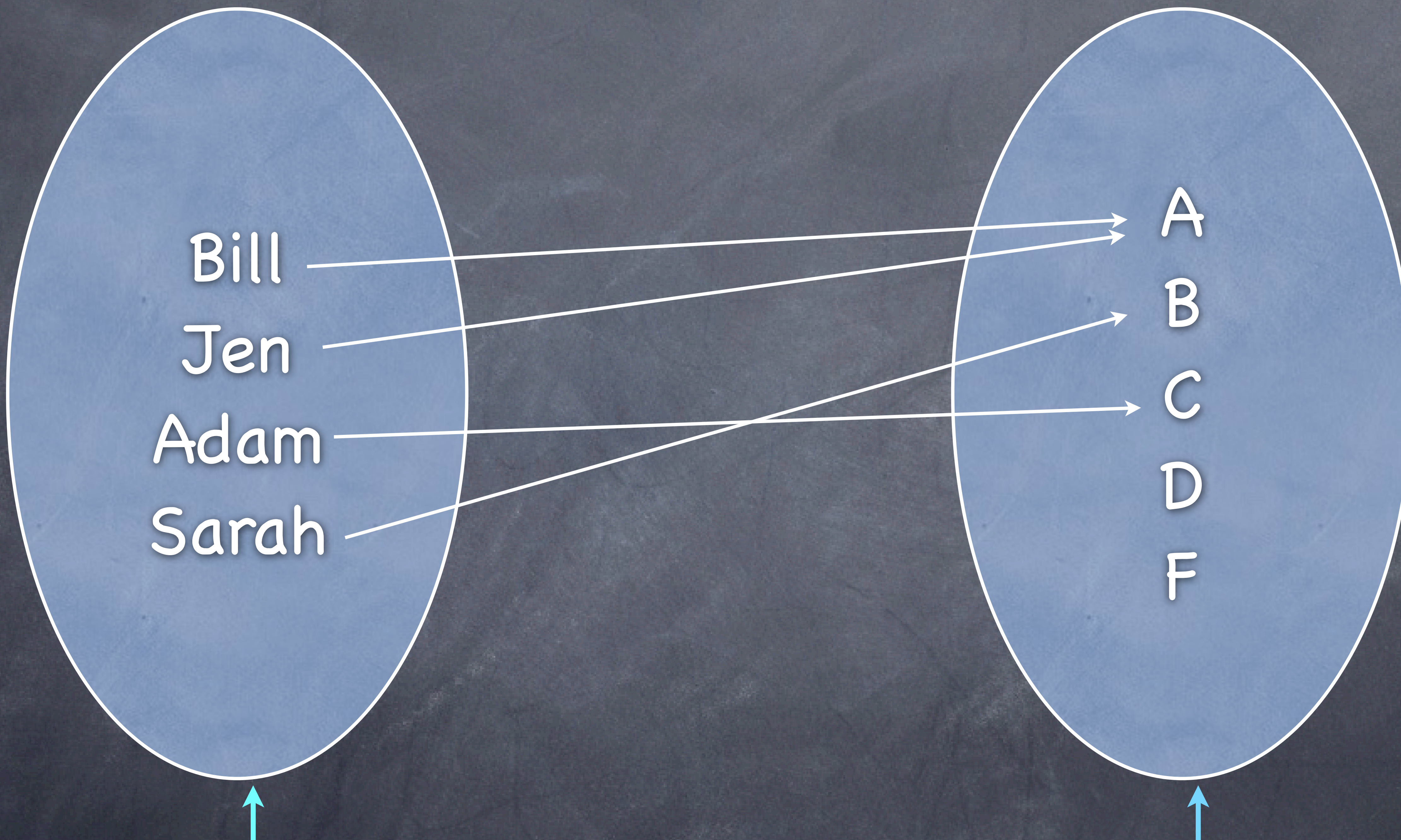
and

$$(\langle x,y_1 \rangle \in f \wedge \langle x,y_2 \rangle \in f) \rightarrow y_1 = y_2$$

Definitions - 2

- For a function $f: A \rightarrow B$
  - A is called the domain
  - B is called the co-domain
- If $f(x)=y$
  - y is called the image of x under f. The set of all images is called the range of f, denoted by $f(A)$
  - x is called a preimage of y

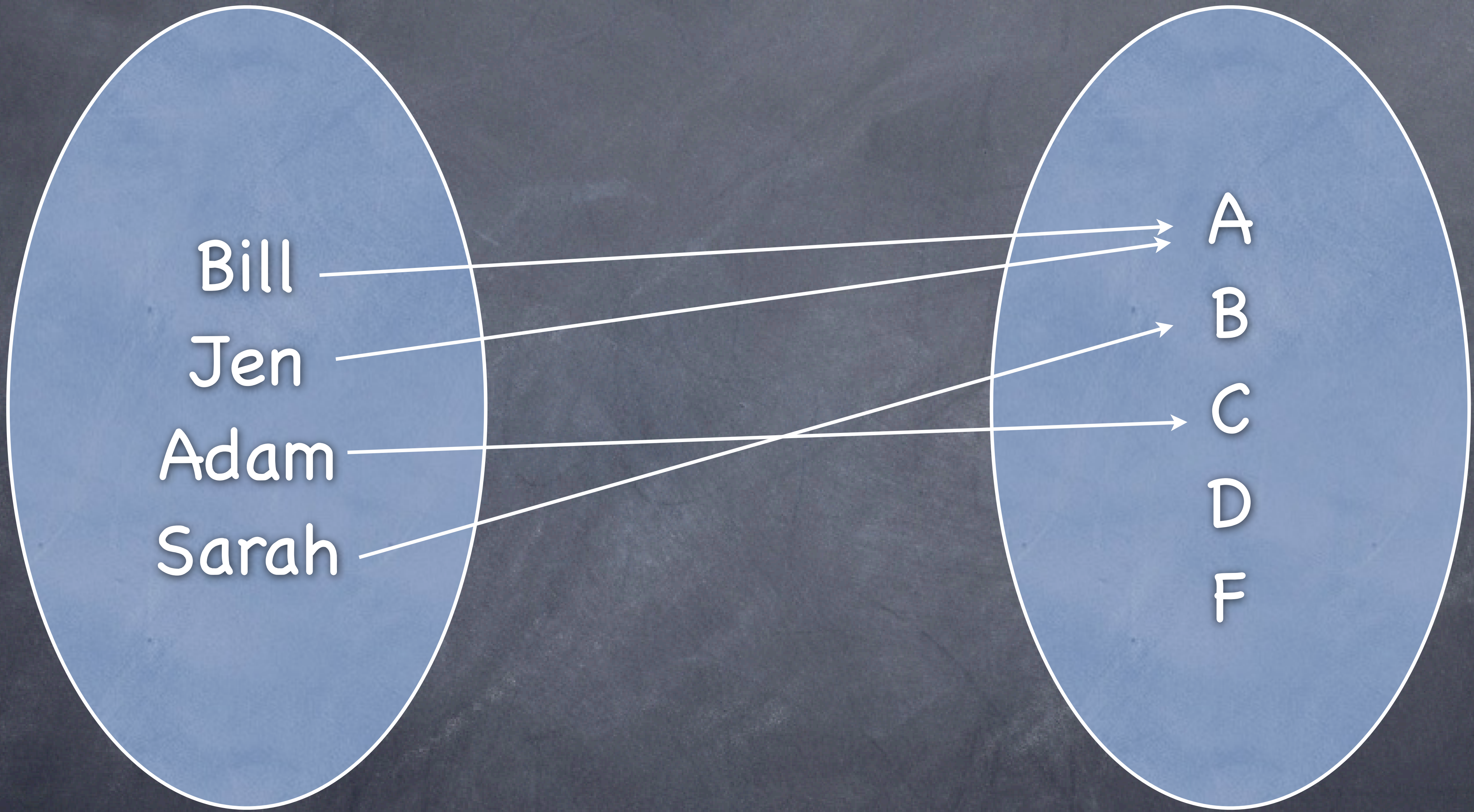
$f: S \rightarrow G$



The domain of f:
S={Bill, Jen, Adam, Sarah}

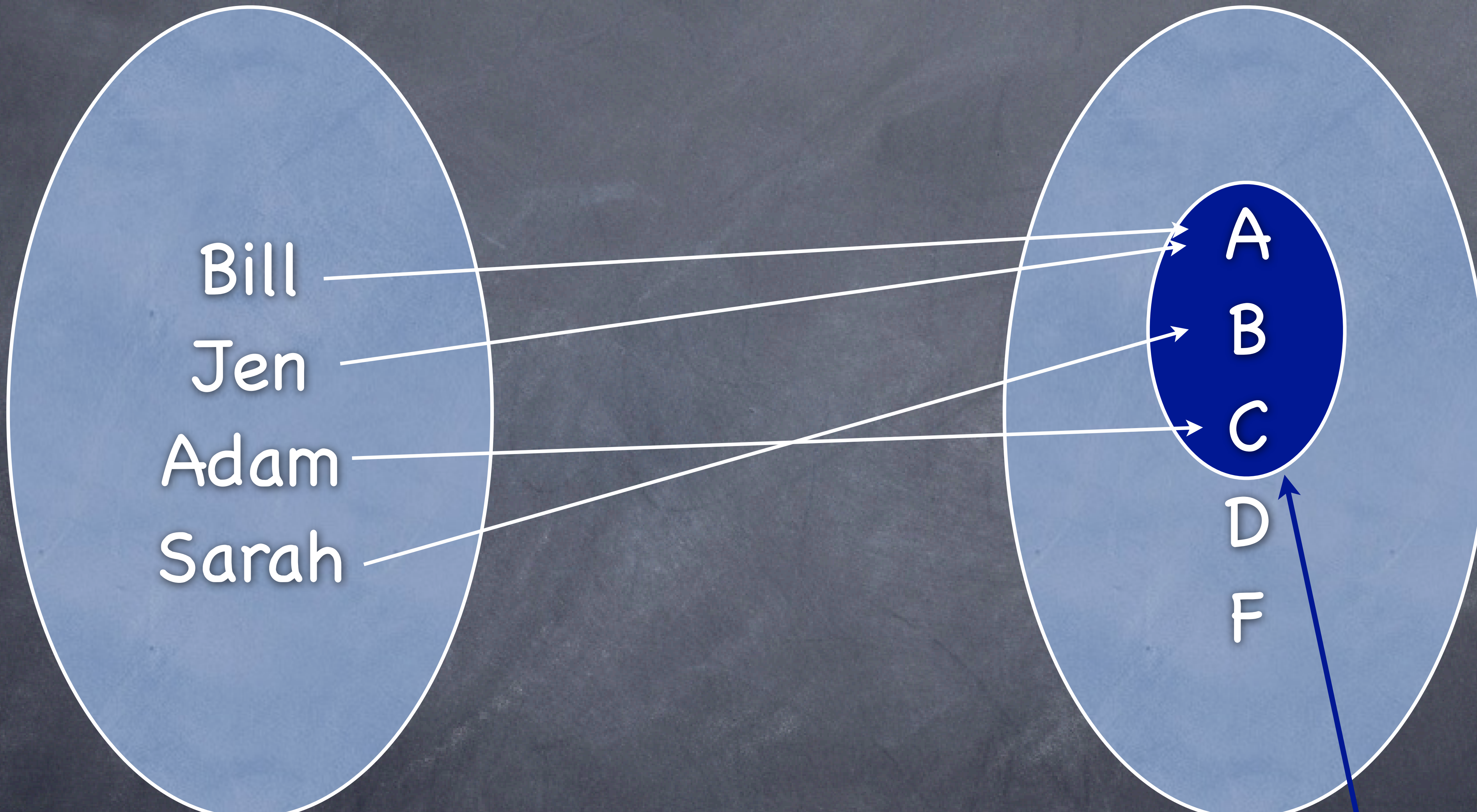
The co-domain of f:
G={A, B, C, D, F}

f: S→G



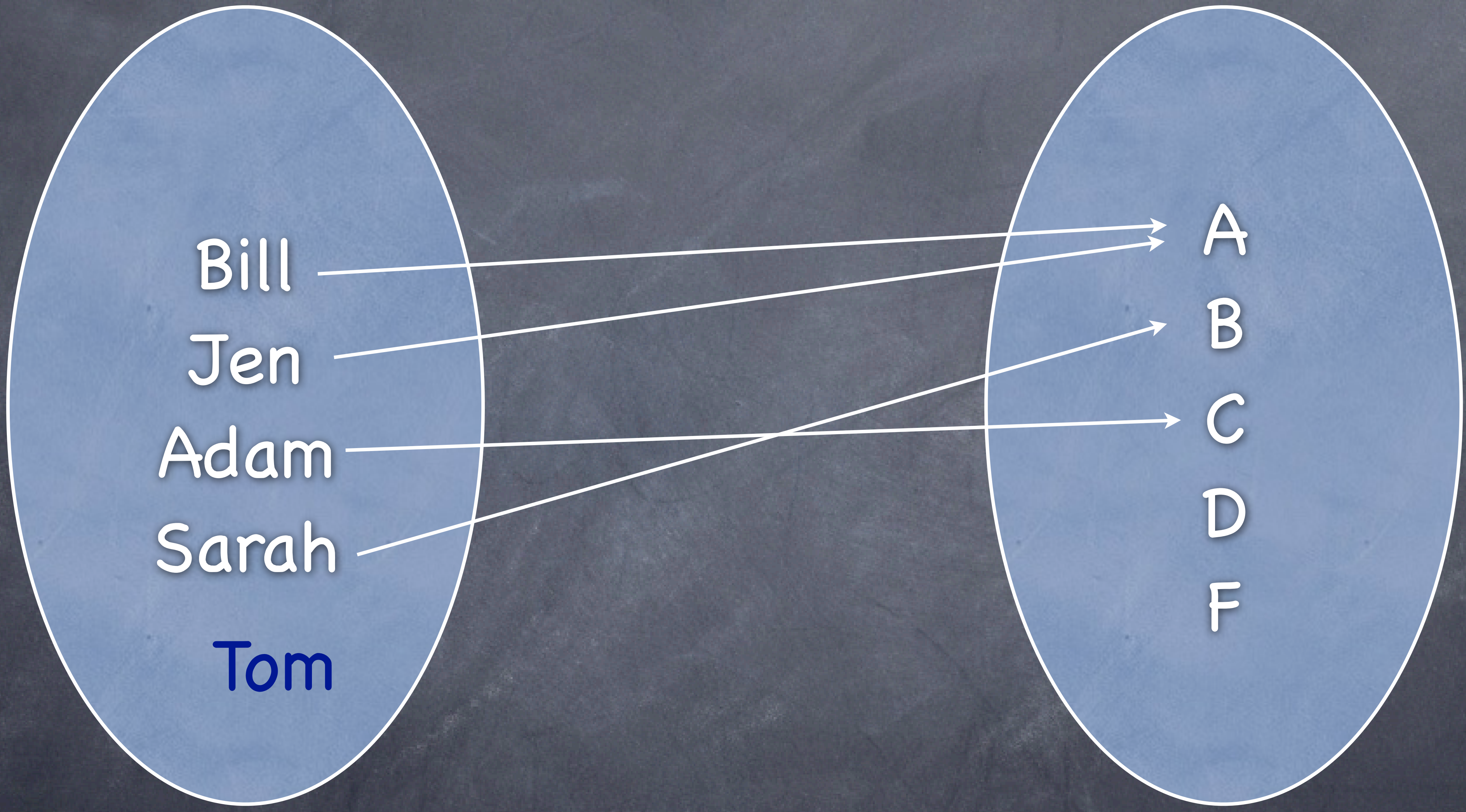
- f(Bill) = A
- The image of Bill is A
- The preimages of A are Bill and Jen

f: S→G



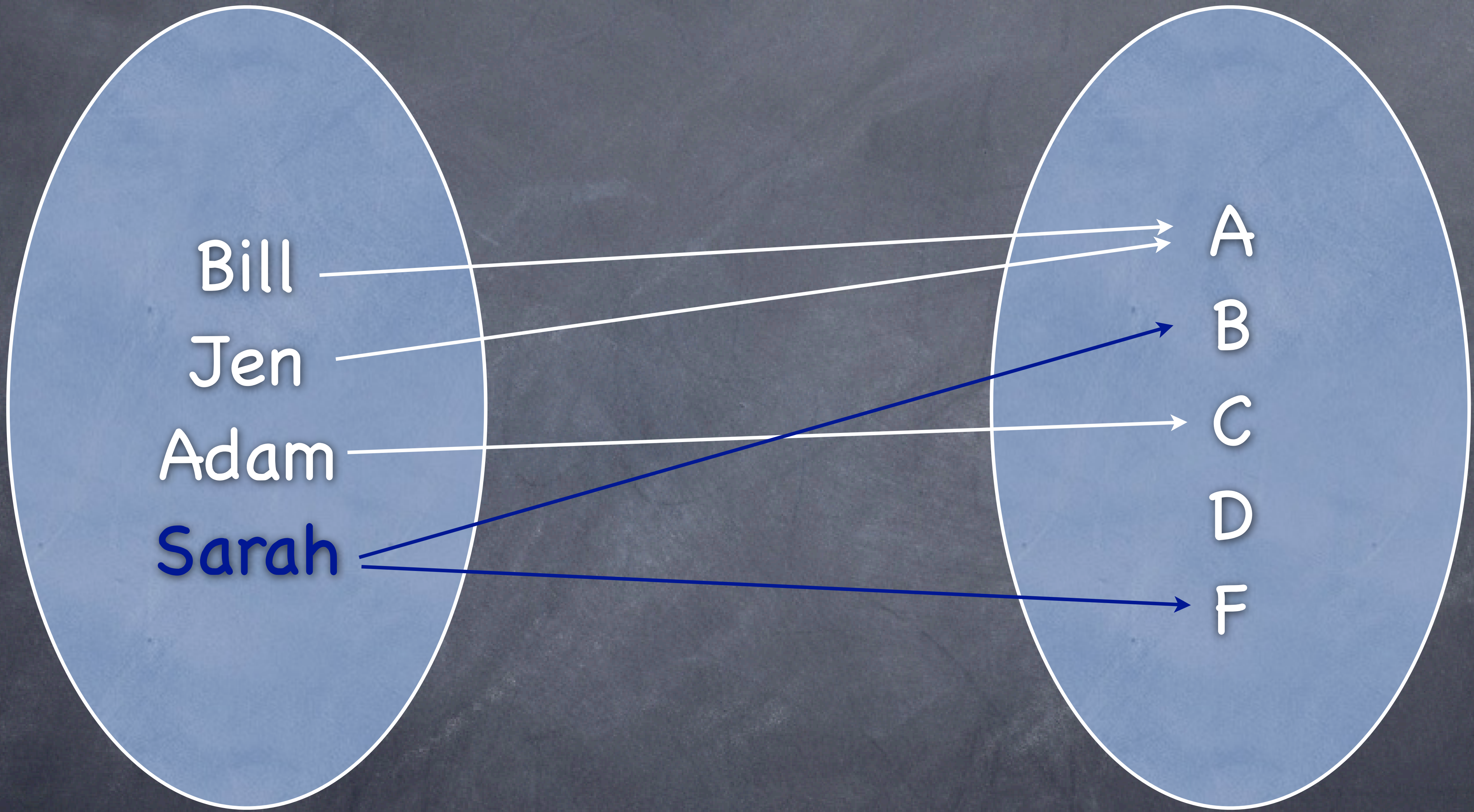
The range of f:
f(S)={A,B,C}

This is NOT a function from S to G



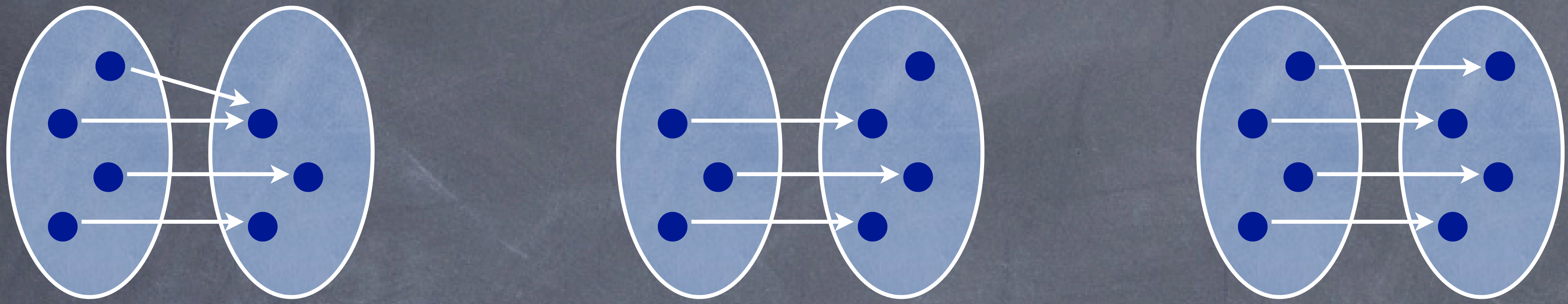
- To be a function, every member of the domain must be mapped to a member of the co-domain

This is NOT a function from S to G



- To be a function, no member of the domain may map to more than one member of the co-domain

Surjections, Injections and Bijections



- f is surjective or onto if its range is equal to its co-domain, i.e. for every y in B there must be an x in A such that f(x)=y.
- f is injective or one-to-one (denoted 1-1), if it maps distinct elements of the domain to distinct elements of the range, i.e. if a≠b then f(a)≠f(b).
- f is bijective or one-to-one correspondence if it is surjective and injective. This means |A| = |B|.

Examples

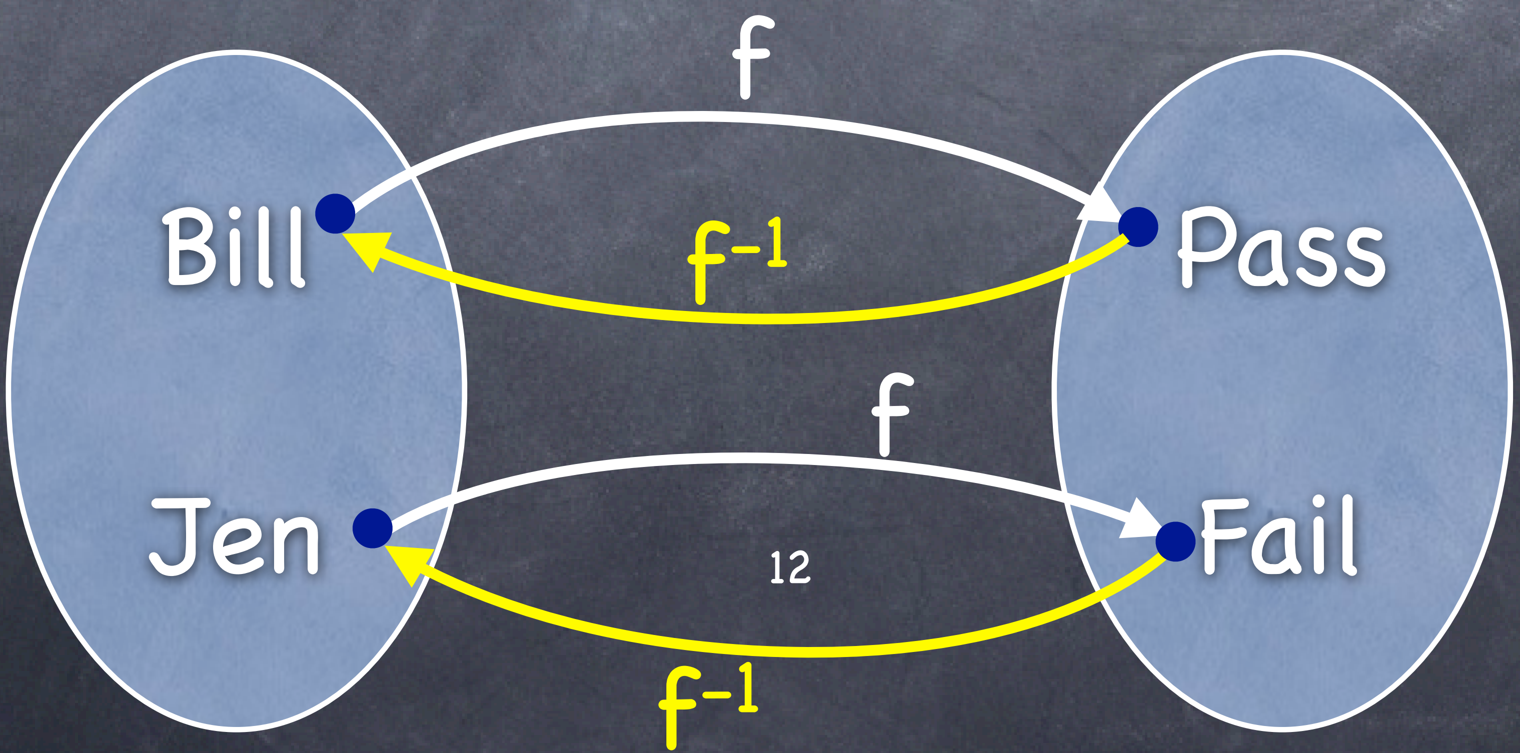
- Let A = B = R, the real numbers. Determine f: A->B, which are injections, surjections, bijections:
  - $f(x) = x$
  - $f(x) = x^2$
  - $f(x) = x^3$
  - $f(x) = |x|$

Inverse Functions

- Let f be a bijection from A to B, then the inverse of f, denoted $f^{-1}:B \to A$ is defined as

$$f^{-1}(y)=x \text{ iff } f(x)=y$$

- A bijective function is called invertible
- A non-bijective function is not invertible



Inverse Functions (example)

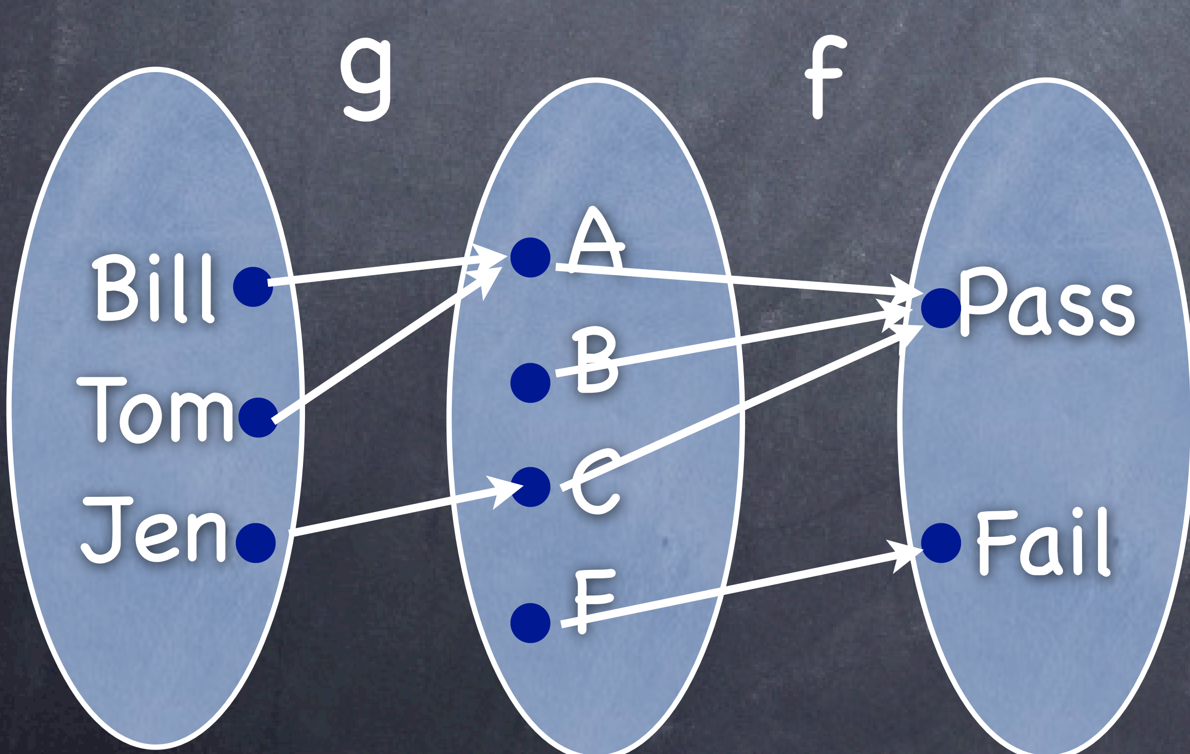
- Determine if the following functions are invertible
  - Let f: {a,b,c}->{1,2,3} be such that f(a)=2, f(b)=1, f(c)=3.
  - Let f: R->R be such that $f(x) = x^2$
  - Let f: R^+->R^+ be such that $f(x) = x^2$

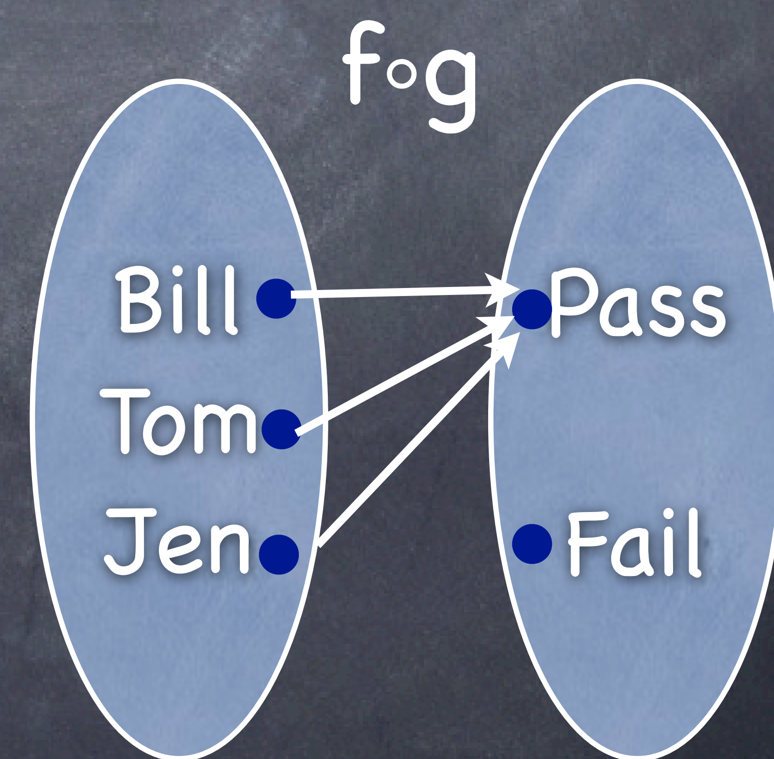
Composition of Functions

- Let $g: A \to B$, $f: B \to C$. The composition of f and g, denoted $f \circ g(x)$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$

- Note that $f \circ g$ is not defined unless the range of g is a subset of the domain of f.





Composition (Example)

Let $f: Z \to Z$ defined by $f(x)=2x+3$ and $g: Z \to Z$ defined by $g(x)=3x+2$. What is $f \circ g(x)$? What is $g \circ f(x)$?

Solution:

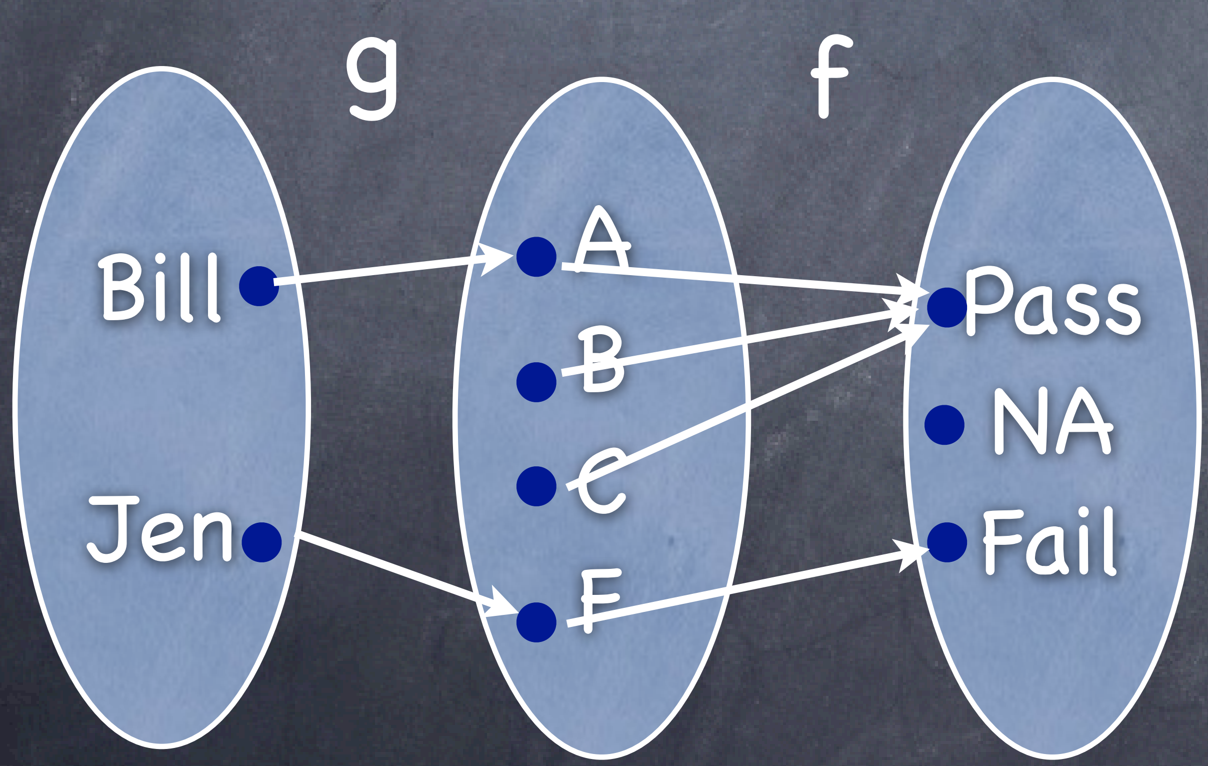
$f \circ g(x)=f(g(x))=f(3x+2)=2(3x+2)+3=6x+7$

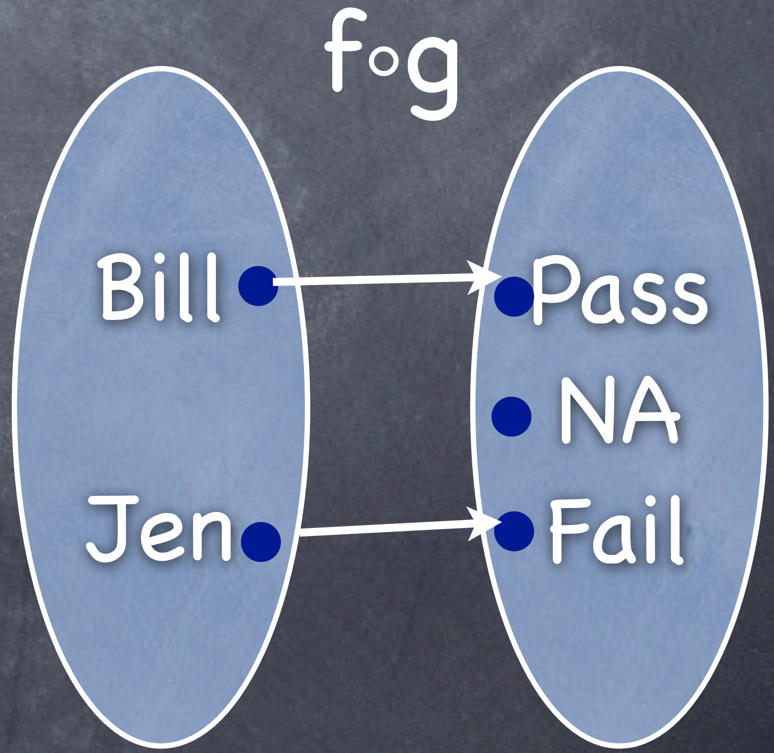
$g \circ f(x)=g(f(x))=g(2x+3)=3(2x+3)+2=6x+11$

Note: $f \circ g(x)$ and $g \circ f(x)$ are not equal

Composition (Example)

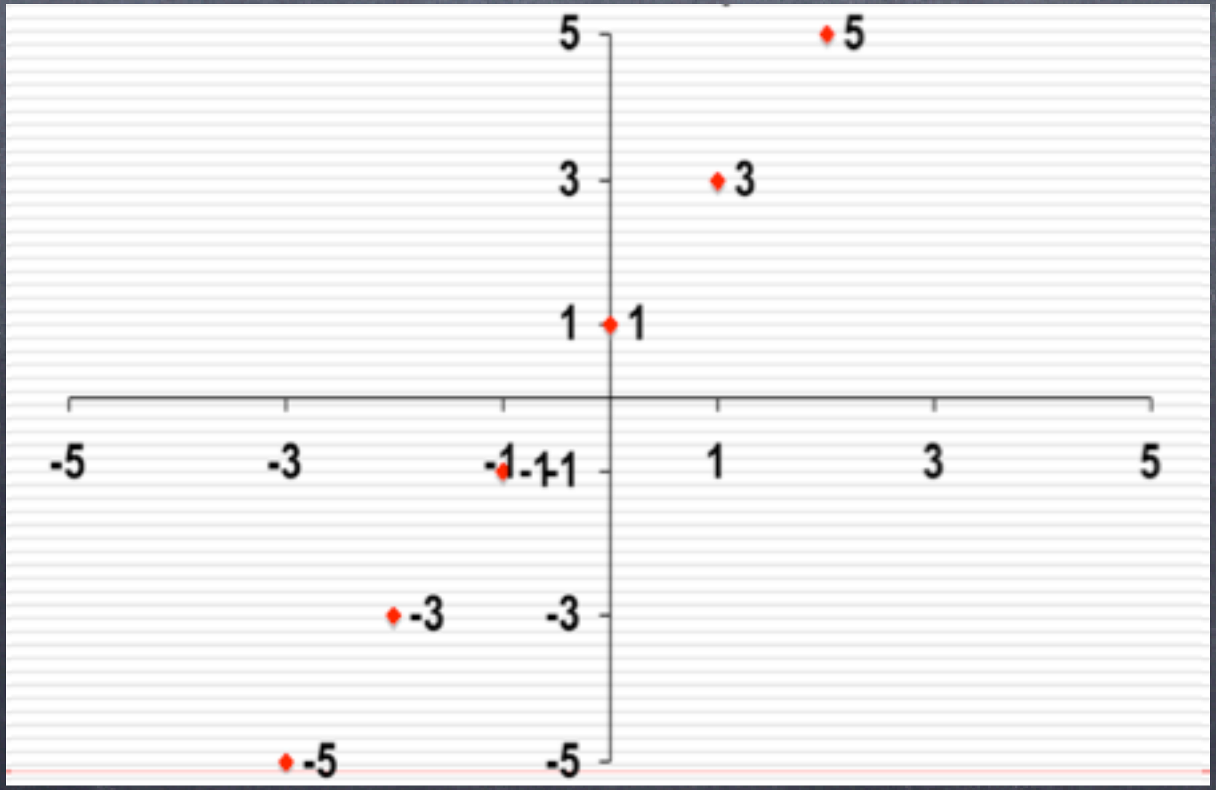
Let f: B→C, g: A→B. If f∘g(x) is injective, what about f and g?





Graphs of Functions

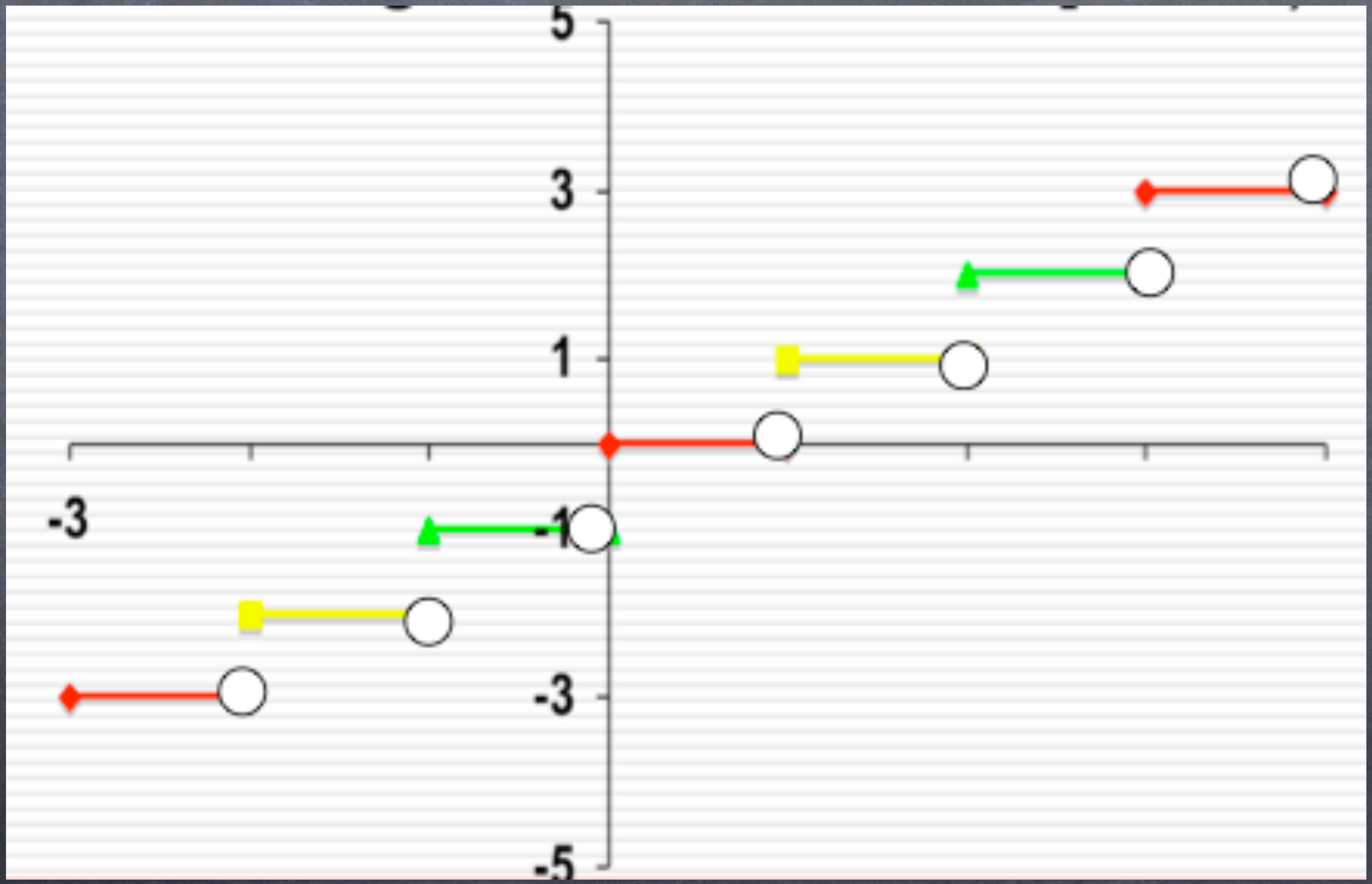
- Let f: A→B. The graph of f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a)=b\}$
- Example: The graph of f:Z->Z where f(x)=2x+1



Important Functions

- Floor function: R->Z, denoted $\lfloor x \rfloor$, is the largest integer less or equal to x
- Ceiling function: R->Z, denoted $\lceil x \rceil$, is the smallest integer greater than or equal to x
- E.g. $\lfloor 3.5 \rfloor = 3$, $\lceil 3.5 \rceil = 4$

Graph of Floor Functions



Show that $\lceil x+n \rceil$ is $\lceil x \rceil + n$ for $x \in R$ and $n \in Z$.

Proof:

Assume $\lceil x \rceil = m$.

$m-1 < x \leq m$

$n+m-1 < x+n \leq m+n$

$\lceil x+n \rceil = m+n = \lceil x \rceil + n$

Q.E.D.

Show that $\lfloor 2x \rfloor$ is $\lfloor x \rfloor + \lfloor x+1/2 \rfloor$ for $x \in R$.

Proof:

Assume $x=n+e$ where $n \in Z$, $e \in R$ and $0 \leq e < 1$.

Case 1: $0 \leq e < 1/2$

$\lfloor 2x \rfloor = \lfloor 2n+2e \rfloor = 2n$ $(0 \leq 2e < 1)$

$\lfloor x \rfloor = \lfloor n+e \rfloor = n$ $(0 \leq e < 1/2)$

$\lfloor x+1/2 \rfloor = \lfloor n+e+1/2 \rfloor = n$ $(1/2 \leq e+1/2 < 1)$

So, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+1/2 \rfloor$

Case 2: $1/2 \leq e < 1$

... See textbook

Reading and Notes

- Read Section 2.3
- Understand the concept of function
- Practice distinguishing injection (1-1), surjection (onto) and bijection
- Practice finding the composition and inverse of functions
- Recommended exercises: 3,5,15,23,45